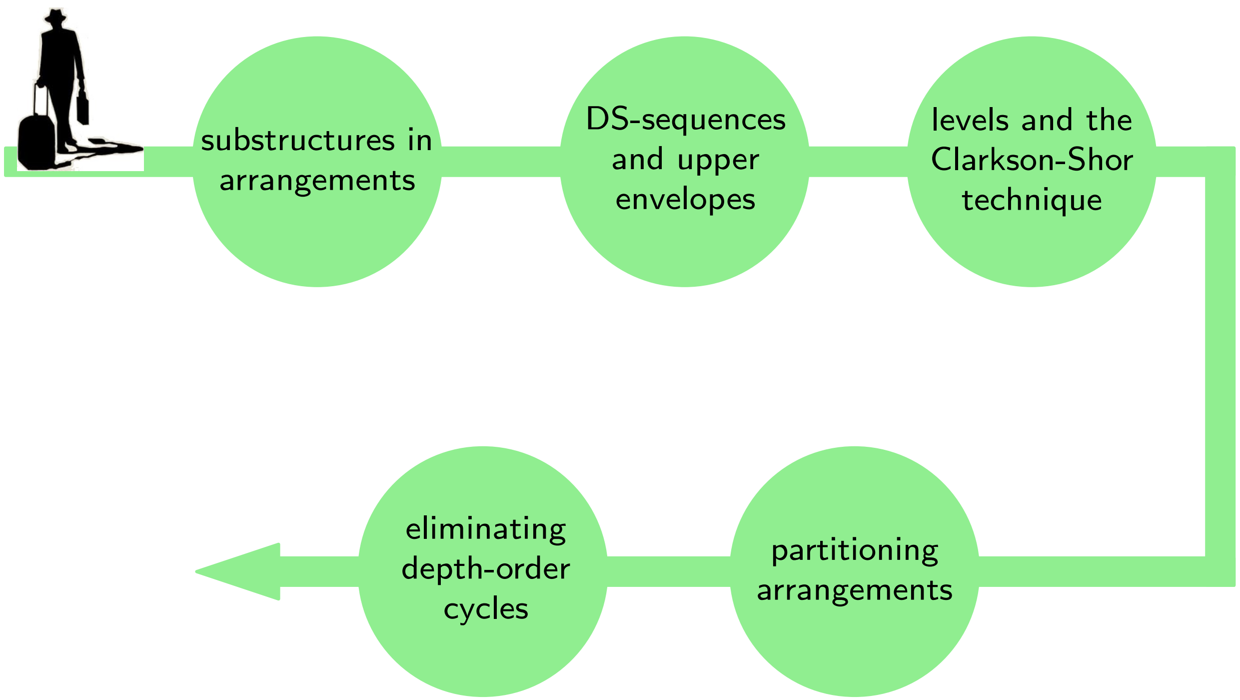
Fundamental Techniques in Computational Geometry:

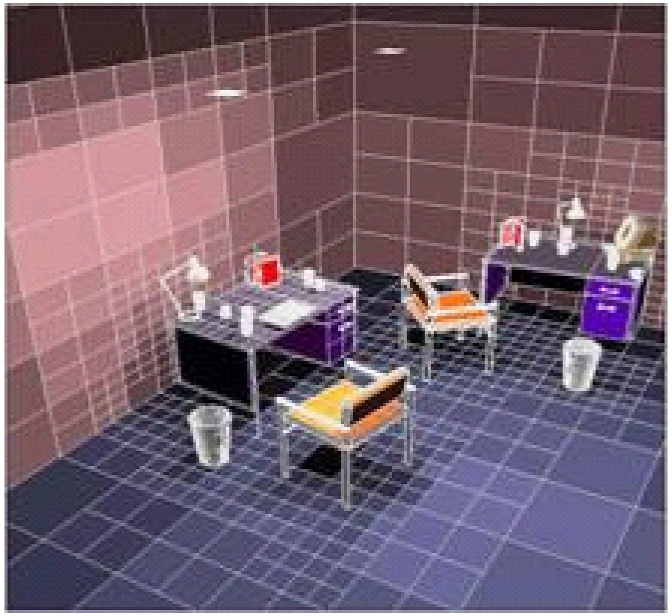
ARRANGEMENTS, PARTITIONS, AND APPLICATIONS

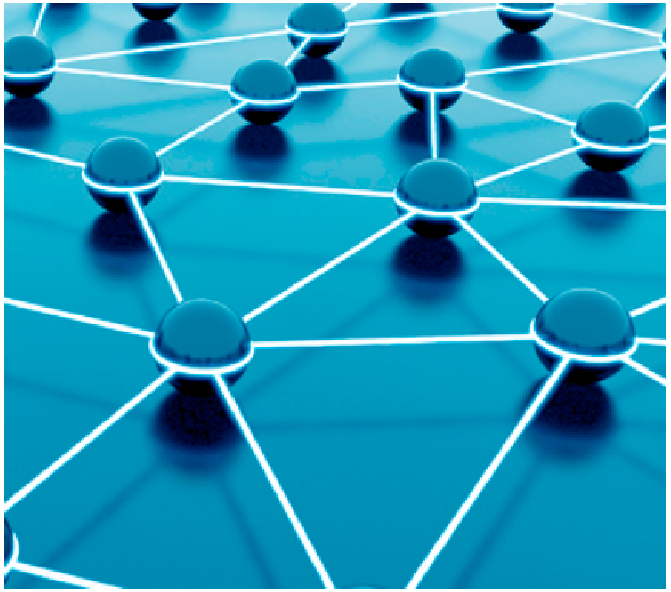


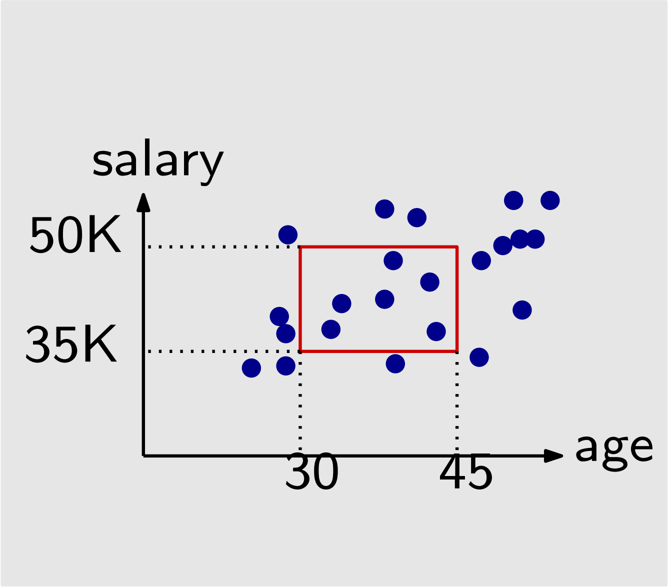
Mark de Berg (TU Eindhoven)

Geometry is everywhere . . .

- geographic information systems
- computer-aided design and manufacturing
- virtual reality
- robotics
- computational biology
- sensor networks
- databases
- and more . . .







Computational Geometry

area within algorithms research dealing with spatial data

- aim for provably correct solutions (no heuristics)
- theoretical analysis of running time, memory usage: $O(\cdots)$

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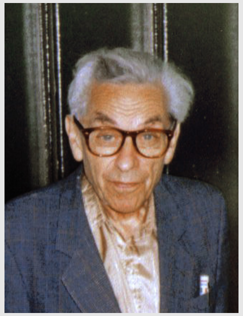
beautiful connections to discrete geometry

- combinatorial bounds needed to analyze algorithms
- computational-geometry techniques useful for combinatorial problems



Paul Erdős

Computational Geometry: Tools of the Trade

Algorithmic design techniques and tools

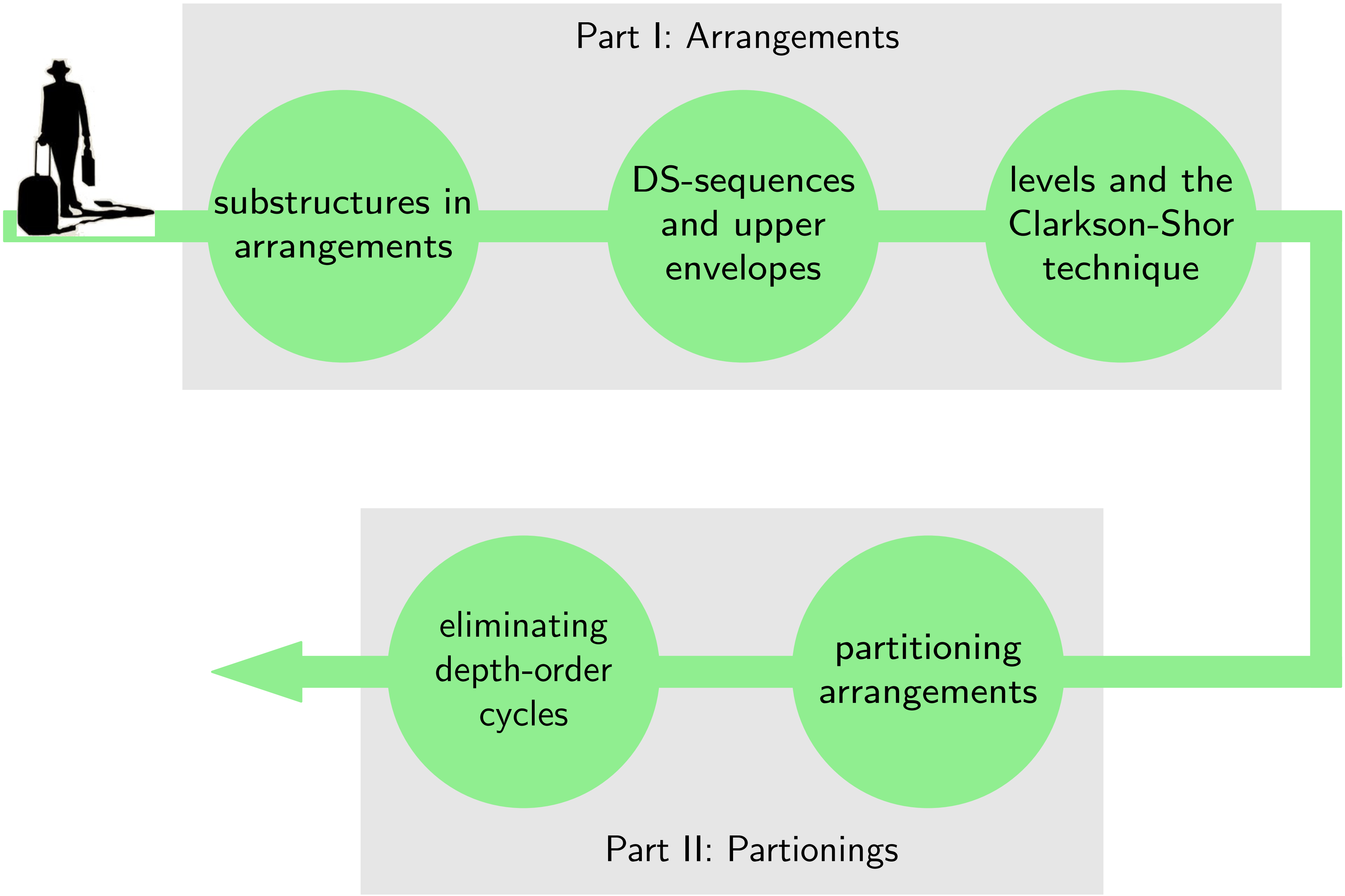
- plane sweep
- geometric divide-and-conquer
- randomized incremental construction
- parametric search
- (multi-level) geometric data structures
- ...

Geometric structures and concepts

- Voronoi diagrams and Delaunay triangulations
- arrangements
- cuttings, simplicial partitions, polynomial partitions
- coresets
- ...

today

Lecture Overview



substructures in arrangements

DS-sequences and upper envelopes

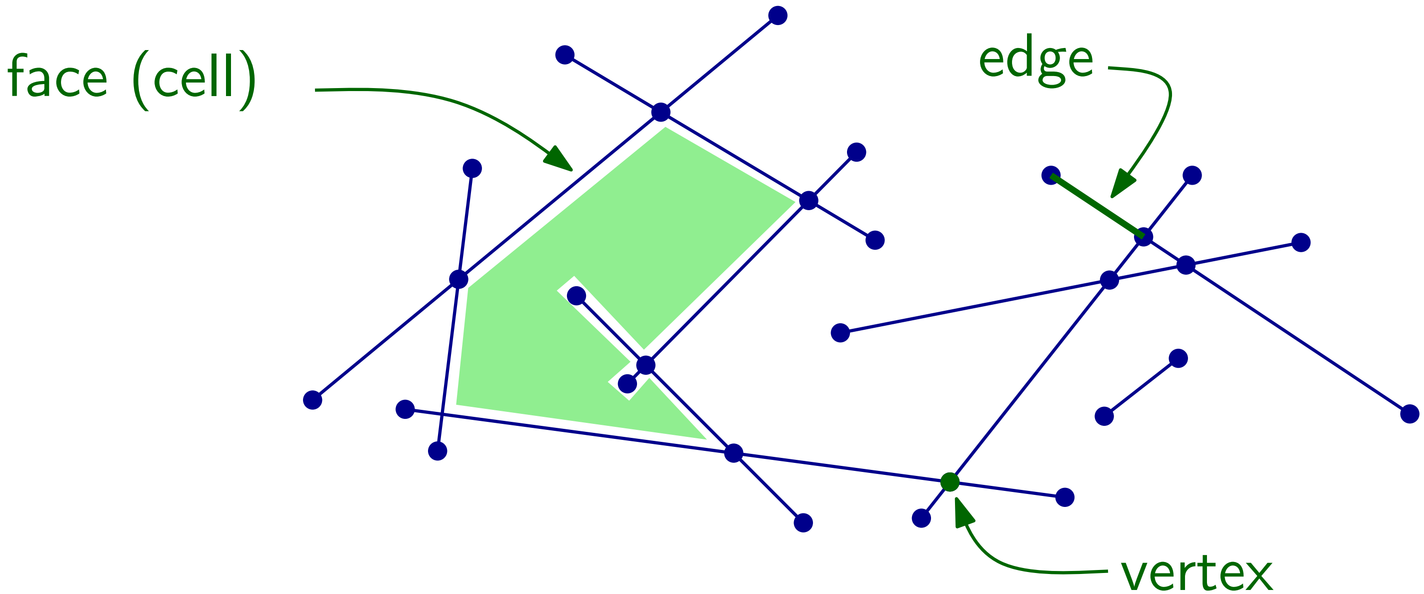
levels and the Clarkson-Shor technique

partitioning arrangements

eliminating depth-order cycles

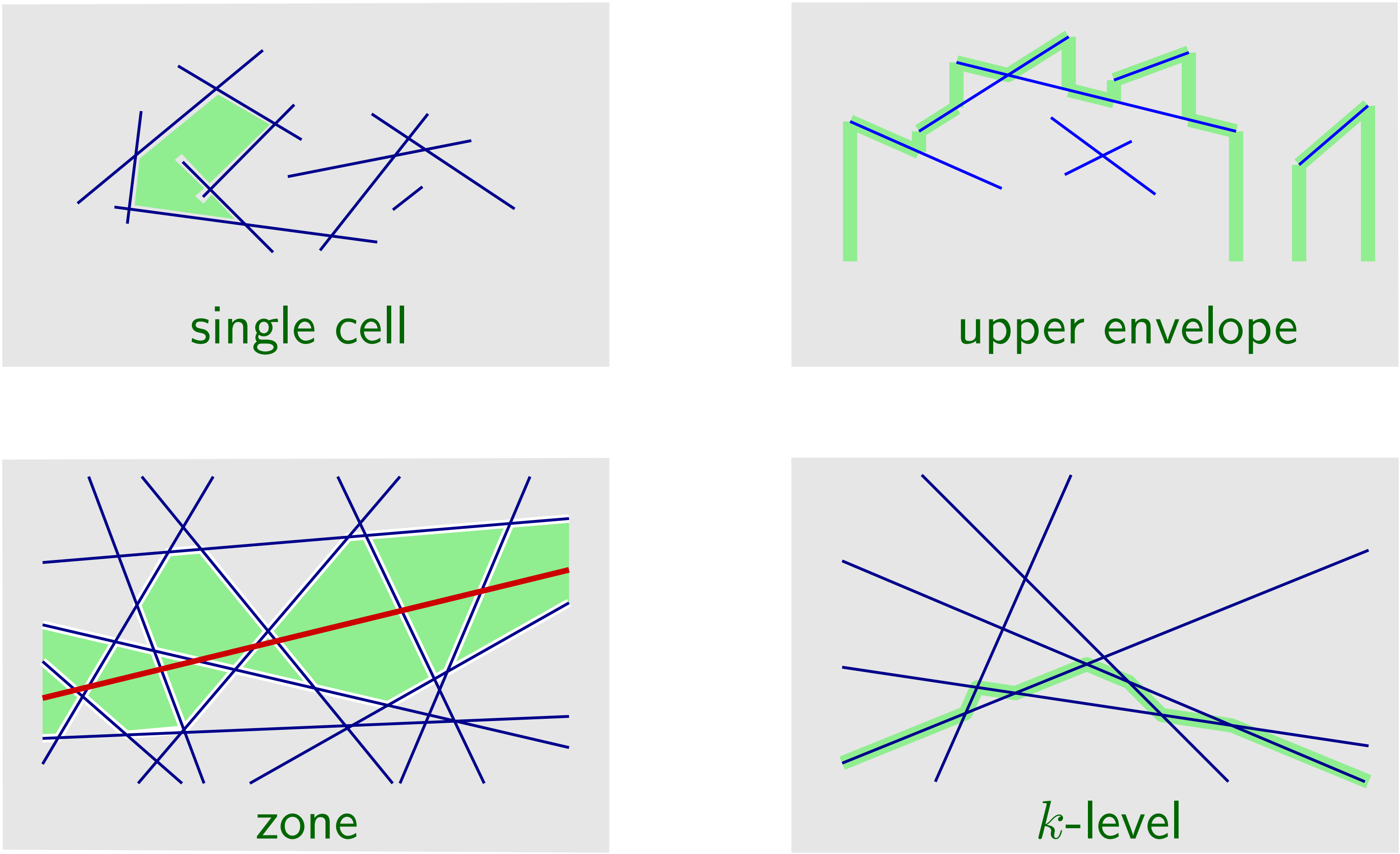
S: set of n lines / segments / curves / etc in $\mathbb{R}^2$

$\mathcal{A}(S)$ = arrangement induced by S
= partitioning of $\mathbb{R}^2$ into faces, edges, and vertices induced by S

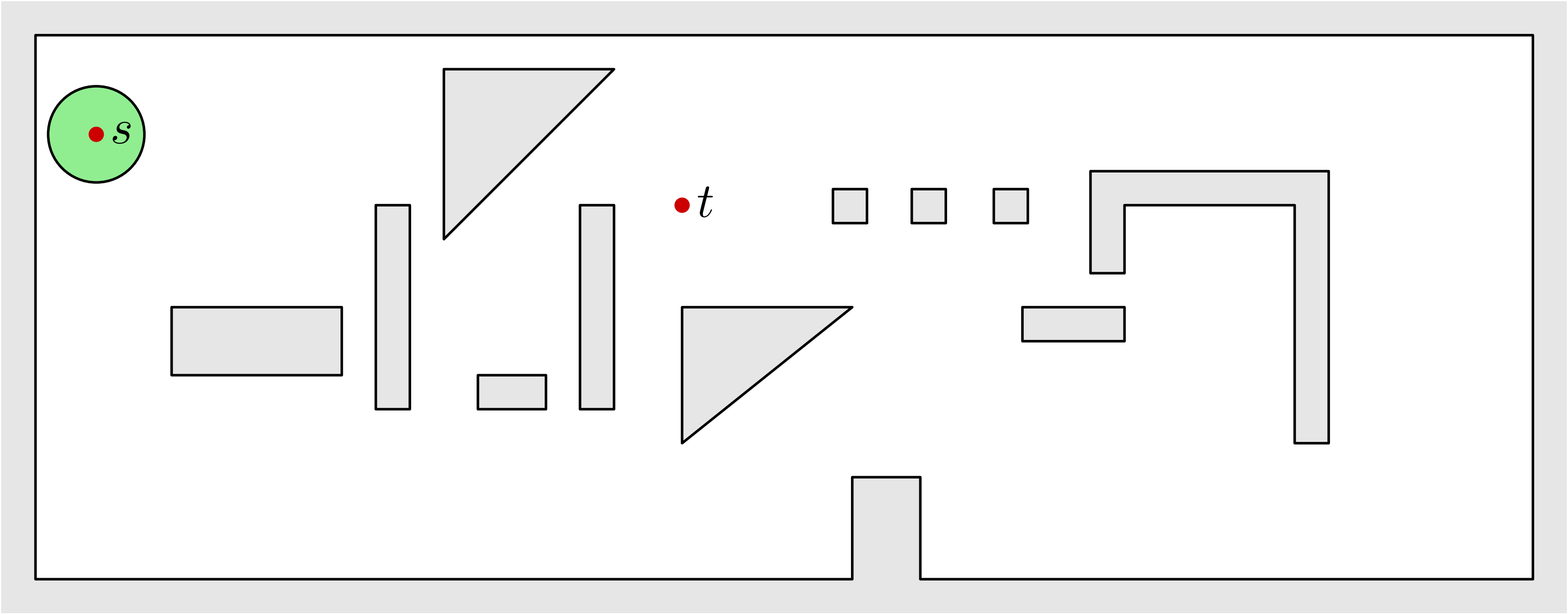


combinatorial complexity of $\mathcal{A}(S)$ = total number of vertices, edges, faces

Substructures in Arrangements



Many geometric problems can be phrased in terms of (substructures in) arrangements by viewing them in an appropriate parametric space.



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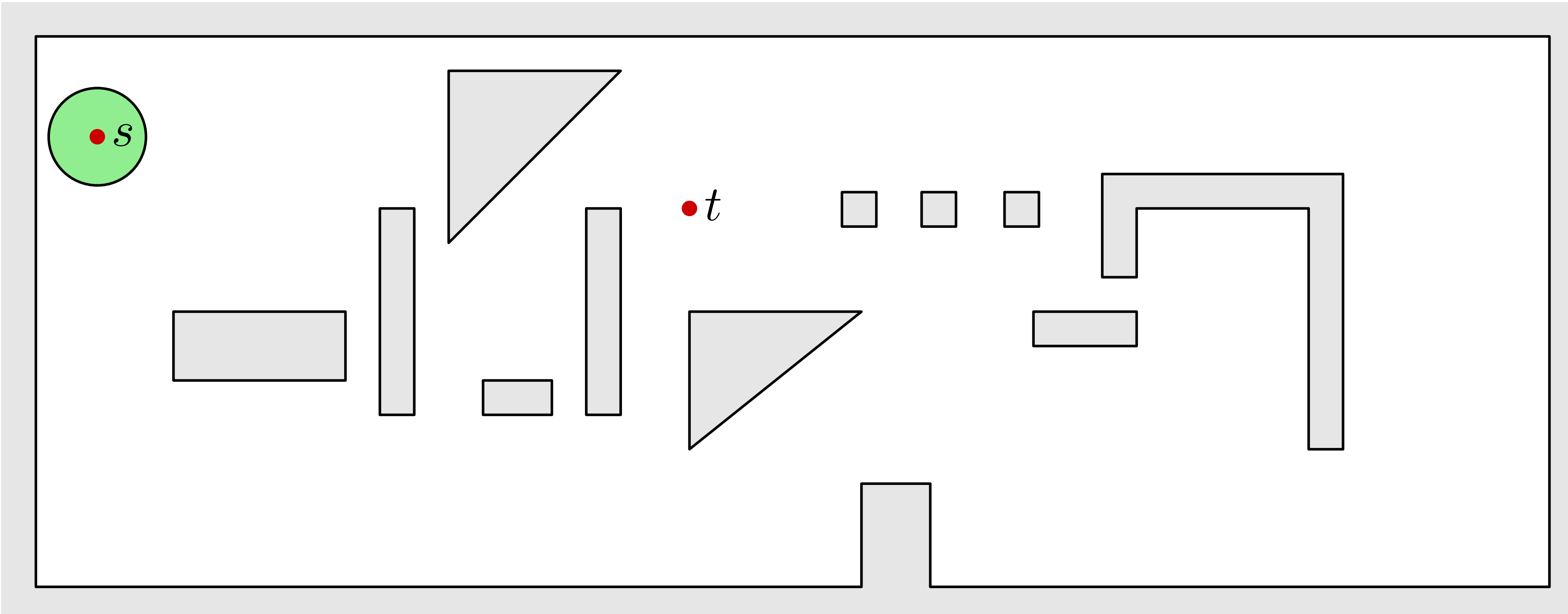
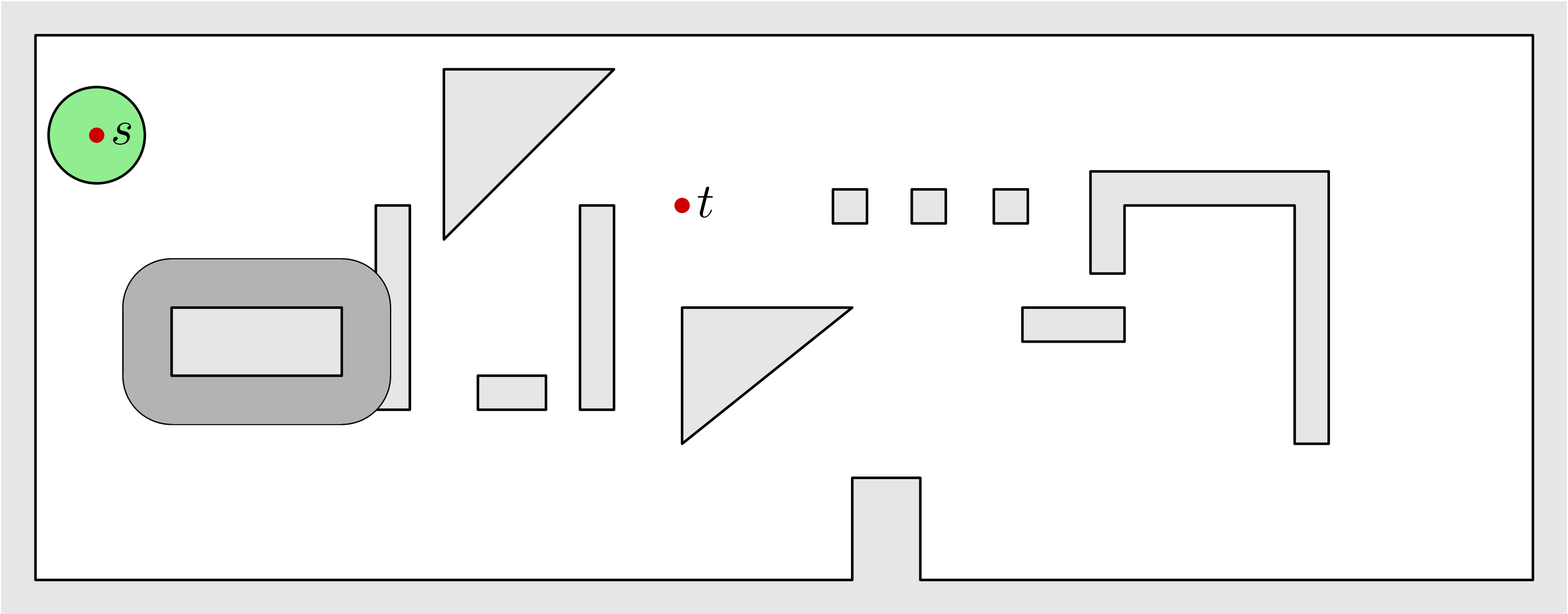


1. Transform problem to motion-planning problem for a point-shaped robot



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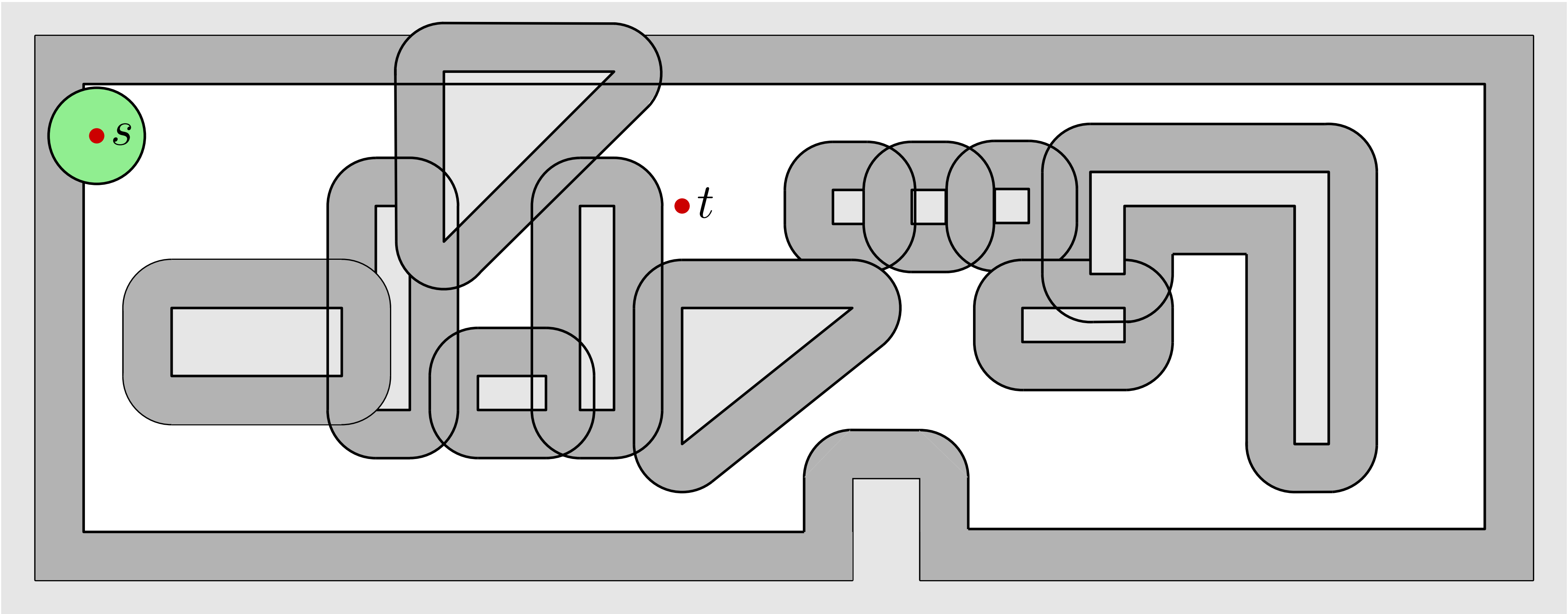

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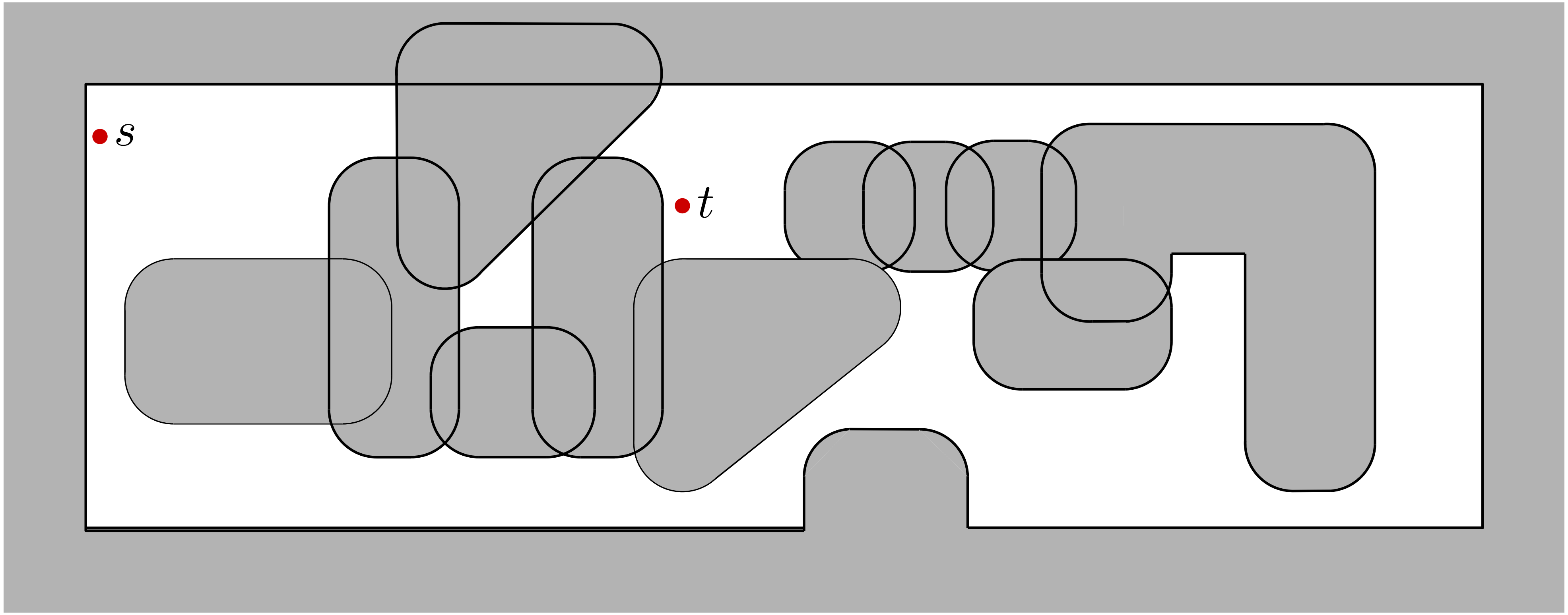

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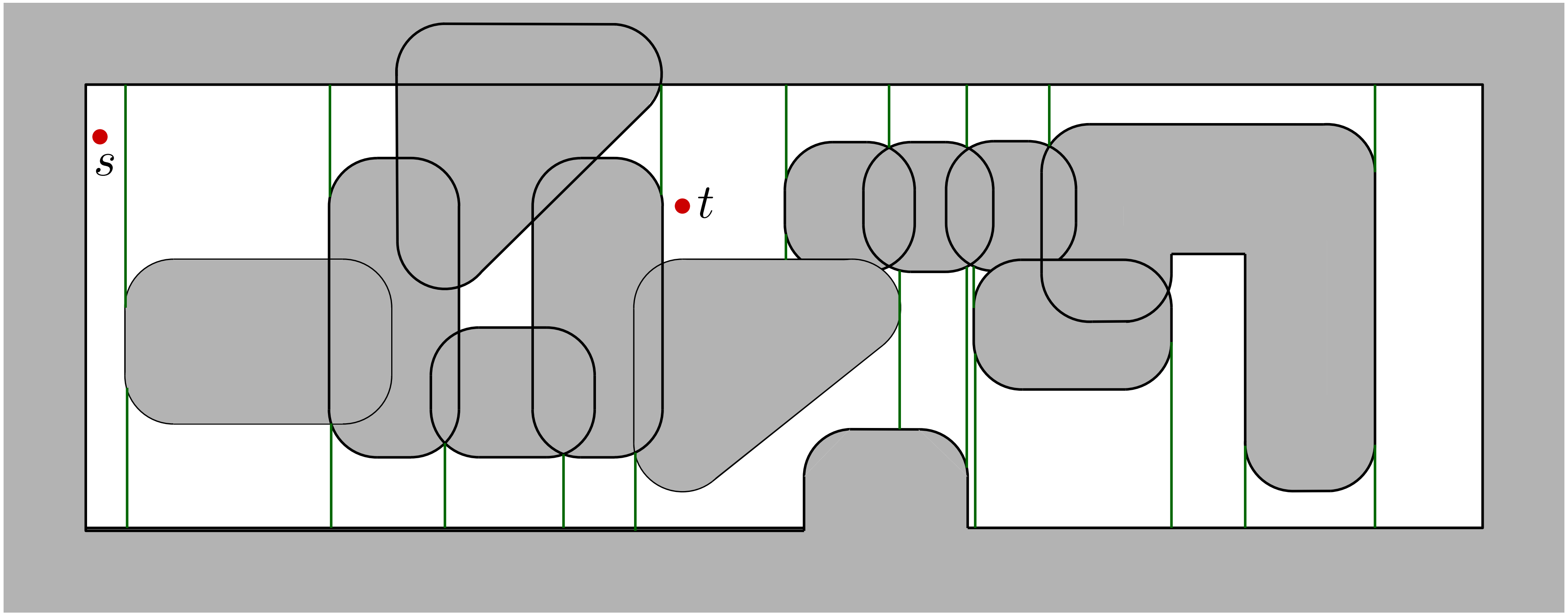
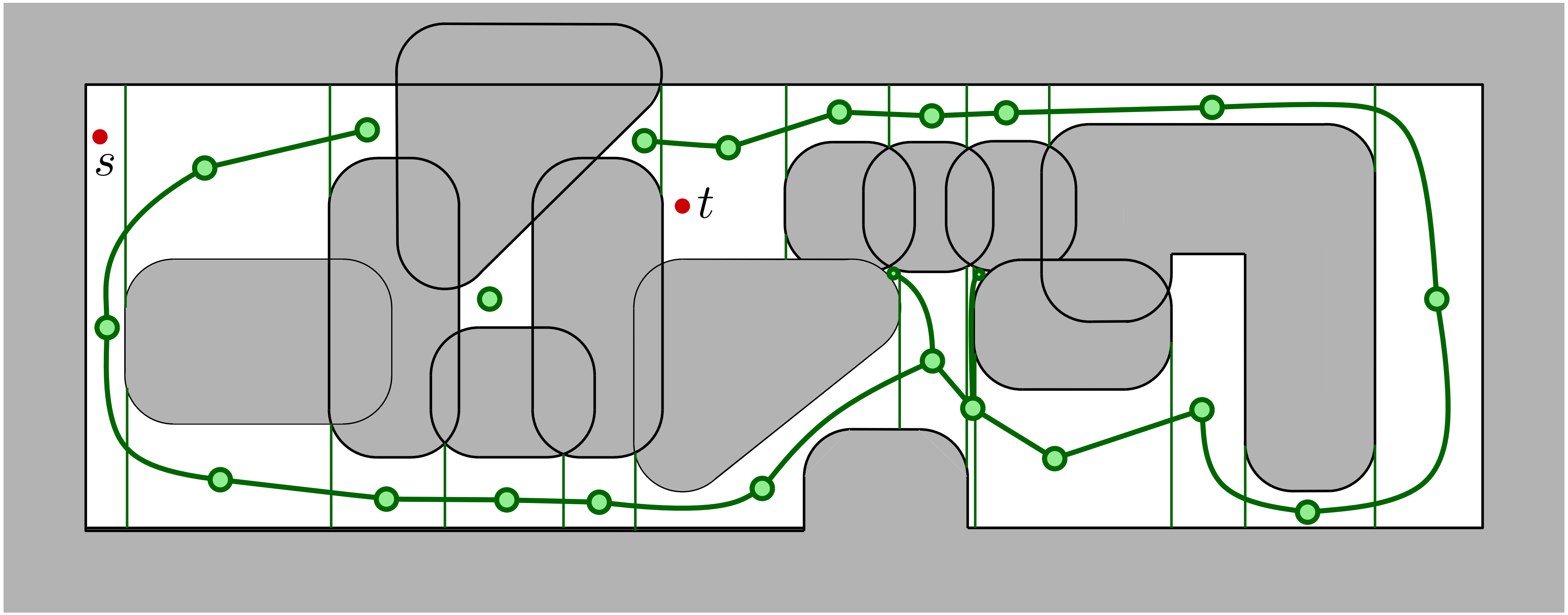
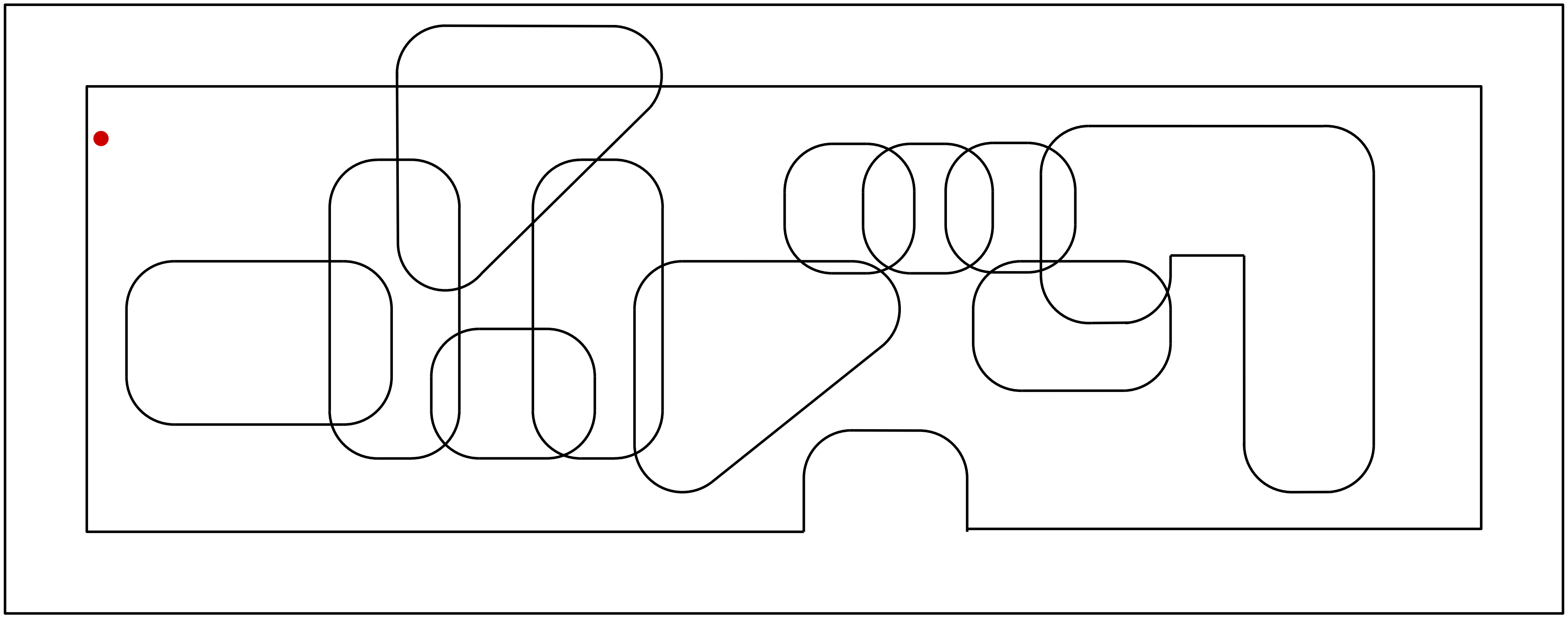


1. Transform problem to motion-planning problem for a point-shaped robot by expanding each obstacle. (Expanded obstacles can intersect!)
2. Decompose free space into "quadrilaterals"

Robot Motion Planning



1. Transform problem to motion-planning problem for a point-shaped robot by expanding each obstacle. (Expanded obstacles can intersect!)
2. Decompose free space into "quadrilaterals"
3. Construct motion graph $\mathcal{G}$ and compute path from s to t in $\mathcal{G}$



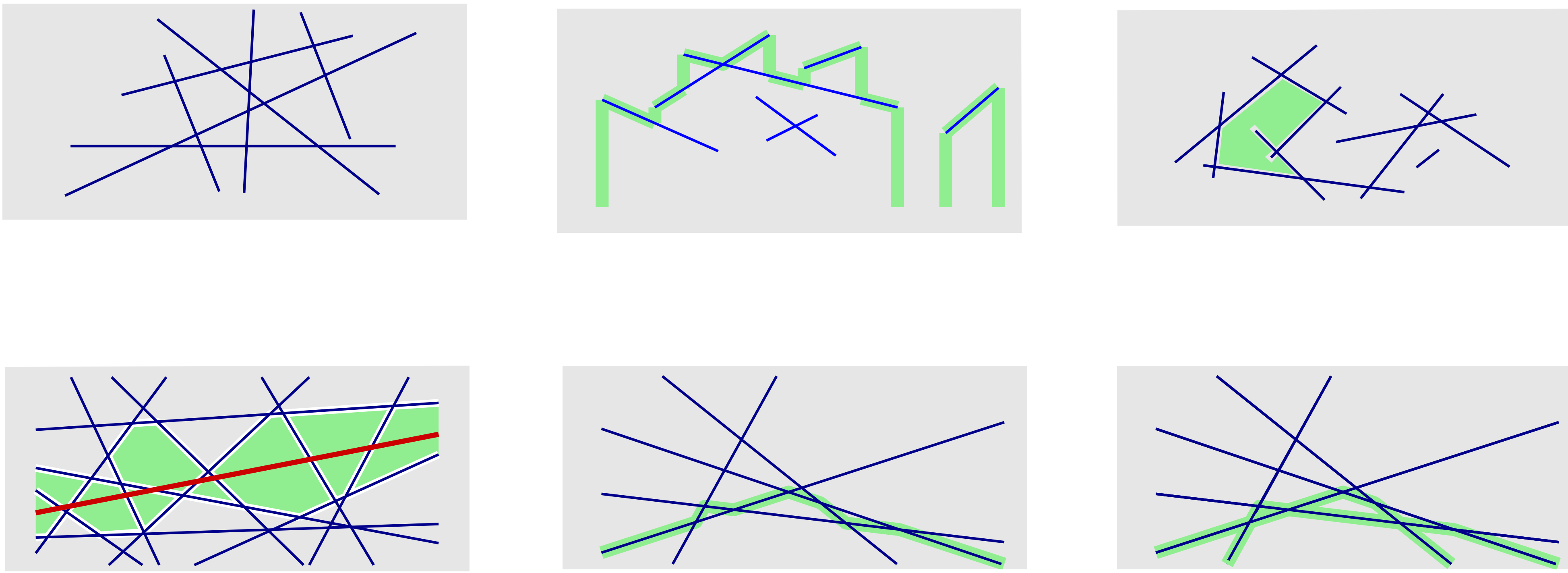
reachable region of the robot

$=$

single cell in arrangement induced by a set S of n curves in $\mathbb{R}^2$

for other types of robots: in $\mathbb{R}^d$, where $d = \#$(degrees of freedom)

The Complexity of (Substructures in) Arrangements



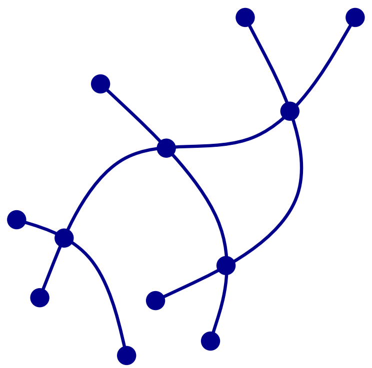
$\mathcal{A}(S)$:= arrangement of set S of n line/segments/curves in $\mathbb{R}^2$

(or: hyperplanes/$(d-1)$-simplices/surface patches in $\mathbb{R}^d$)

what is the worst-case complexity of these substructures in$\mathcal{A}(S)$?

The Complexity of Arrangements

Theorem. Let S be a set of n simple curves such that any two curves intersect at most s times, where S is a fixed constant. Then the complexity of the full arrangement $\mathcal{A}(S)$ is $O(n^2)$.



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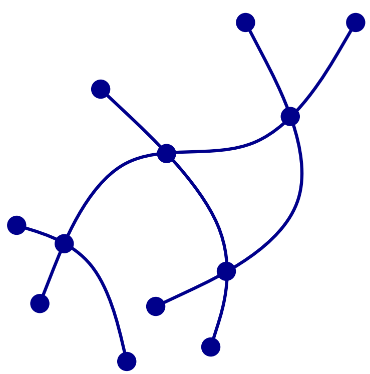

Assume curves are finite.

- number of vertices
- number of edges
- number of faces

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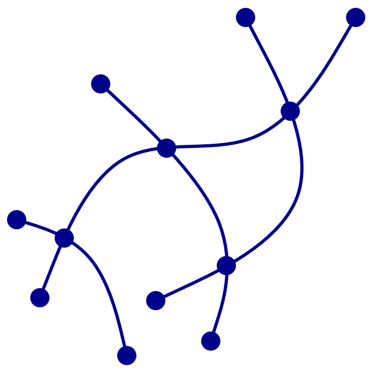

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- number of vertices $\quad |V| \leqslant 2n + s \cdot \binom{n}{2} = O(n^2)$
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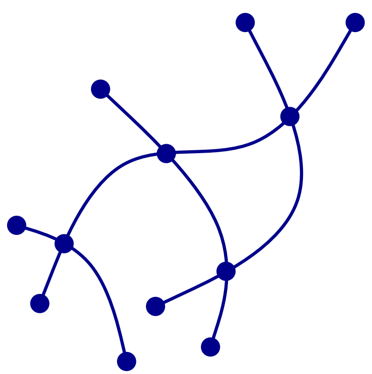

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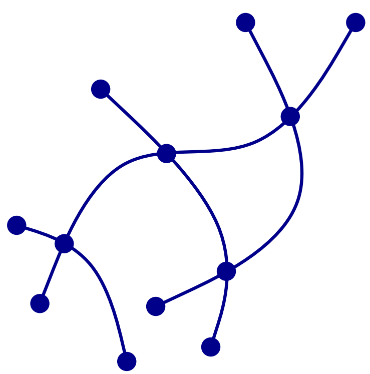

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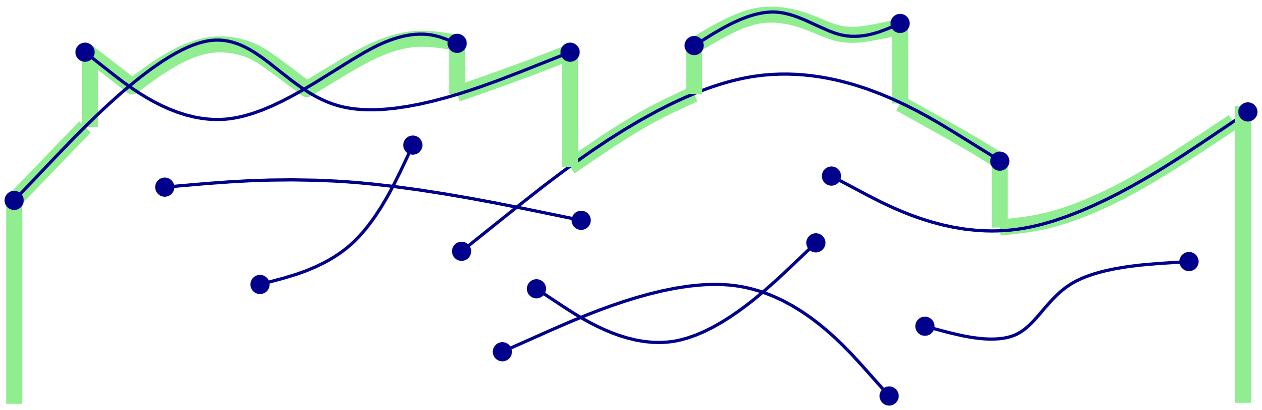
Euler's formula:

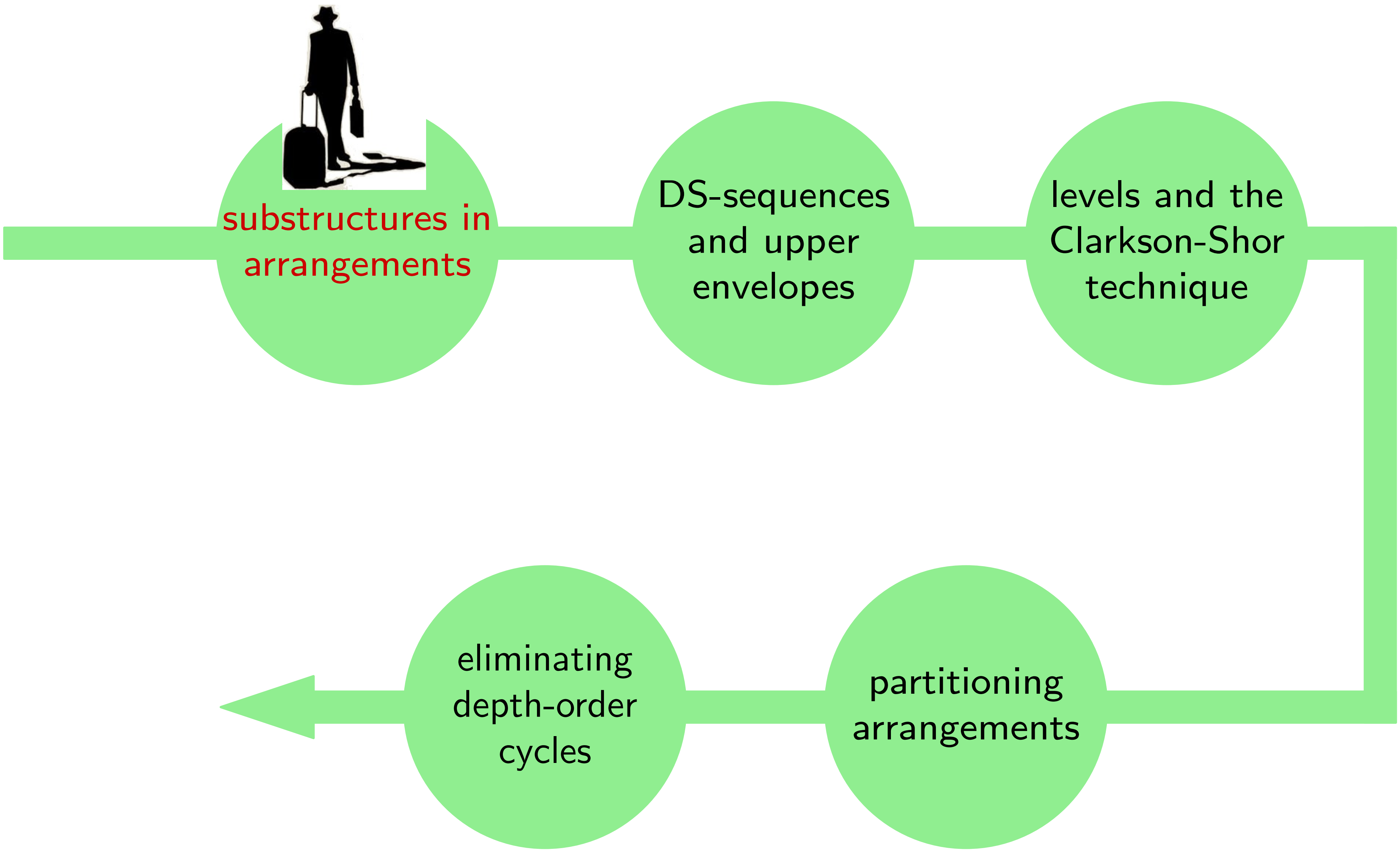
$$|V| - |E| + |F| = 2$$

∎

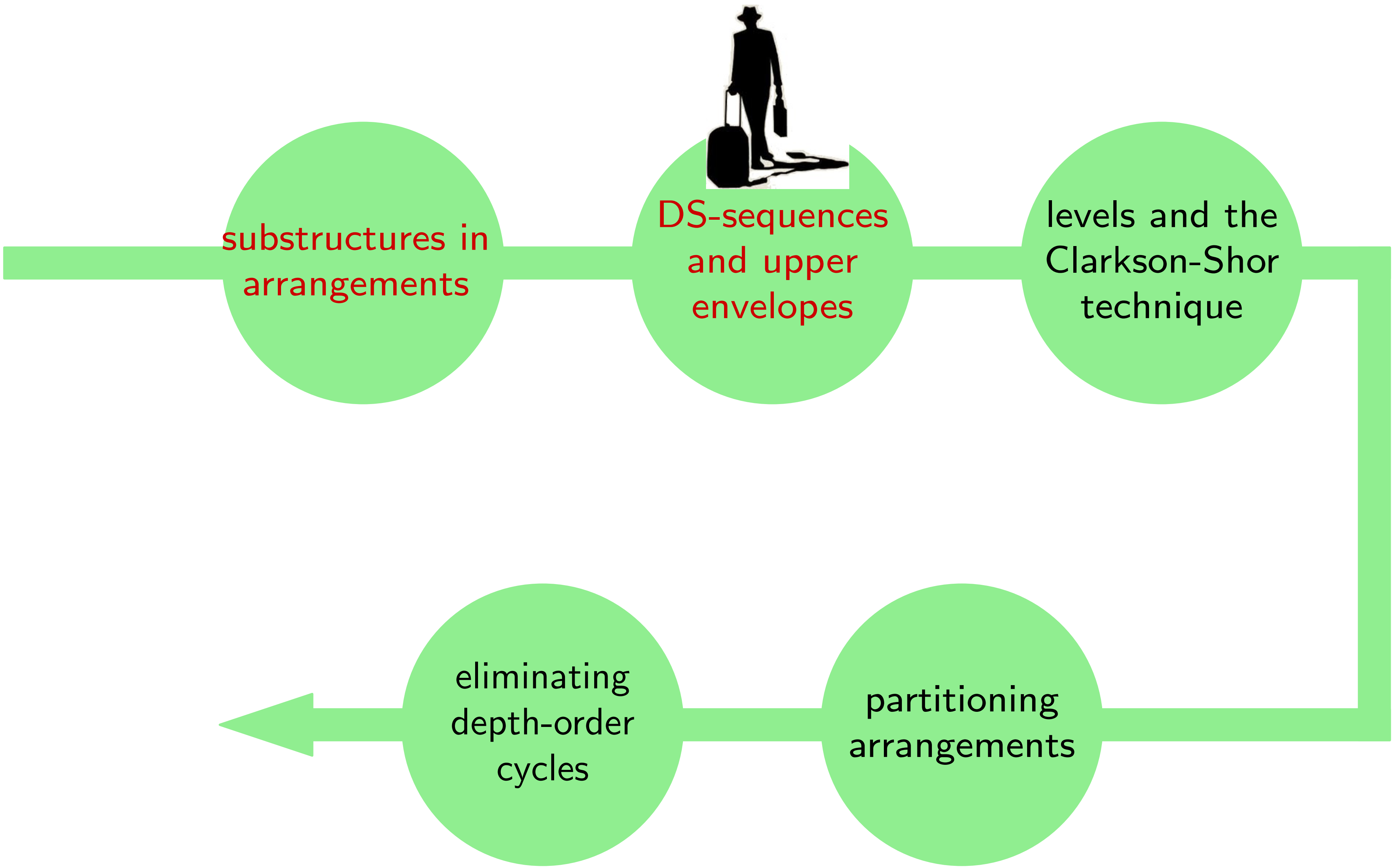


analysis using Davenport-Schinzel sequences

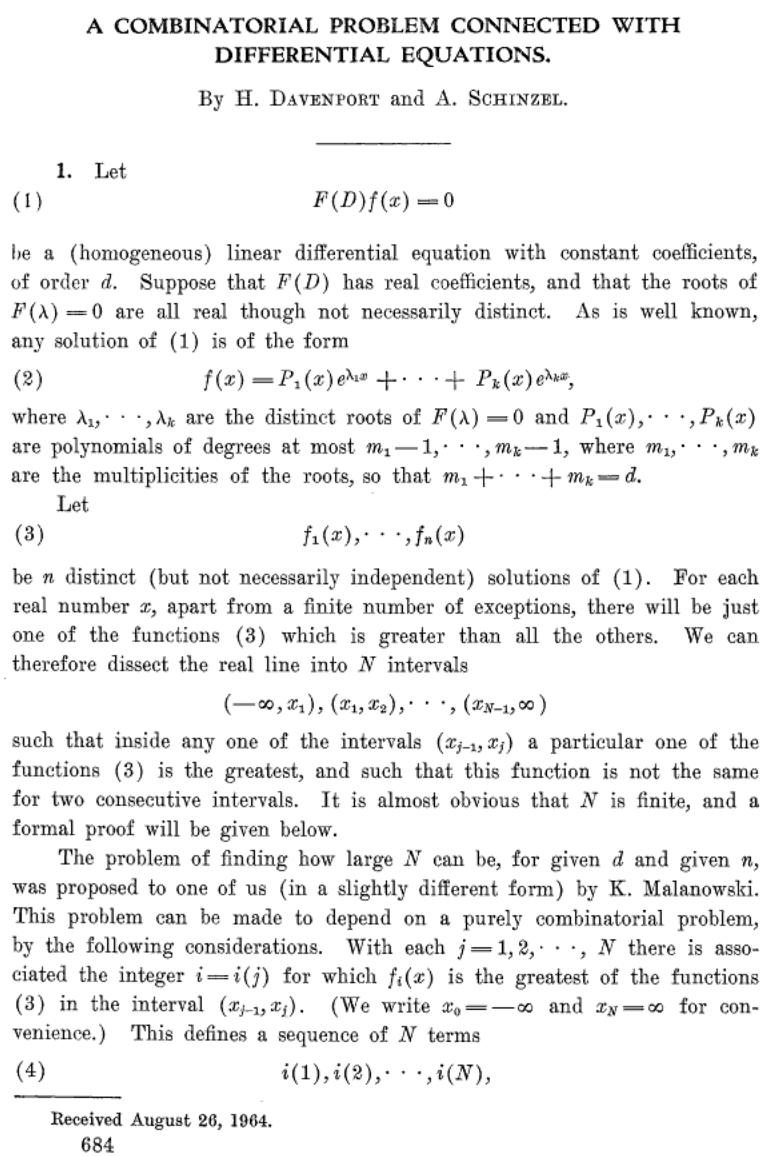
Lecture Overview



Lecture Overview



Davenport-Schinzel sequences



A COMBINATORIAL PROBLEM CONNECTED WITH DIFFERENTIAL EQUATIONS.

By H. Davenport and A. Schinzel.

1. Let

(1) $$F(D)f(x)=0$$

be a (homogeneous) linear differential equation with constant coefficients, of order d. Suppose that $F(D)$ has real coefficients, and that the roots of $F(\lambda)=0$ are all real though not necessarily distinct. As is well known, any solution of (1) is of the form

(2) $$f(x)=P_1(x)e^{\lambda_1 x}+\cdots+P_k(x)e^{\lambda_k x},$$

where $\lambda_1,\cdots,\lambda_k$ are the distinct roots of $F(\lambda)=0$ and $P_1(x),\cdots,P_k(x)$ are polynomials of degrees at most $m_1-1,\cdots,m_k-1$, where $m_1,\cdots,m_k$ are the multiplicities of the roots, so that $m_1+\cdots+m_k=d$.

Let

(3) $$f_1(x),\cdots,f_n(x)$$

be n distinct (but not necessarily independent) solutions of (1). For each real number x, apart from a finite number of exceptions, there will be just one of the functions (3) which is greater than all the others. We can therefore dissect the real line into N intervals

$$(-\infty,x_1),(x_1,x_2),\cdots,(x_{N-1},\infty)$$

such that inside any one of the intervals (x_{j-1},x_j) a particular one of the functions (3) is the greatest, and such that this function is not the same for two consecutive intervals. It is almost obvious that N is finite, and a formal proof will be given below.

The problem of finding how large N can be, for given d and given n, was proposed to one of us (in a slightly different form) by K. Malanowski. This problem can be made to depend on a purely combinatorial problem, by the following considerations. With each $j=1,2,\cdots,N$ there is associated the integer $i=i(j)$ for which $f_i(x)$ is the greatest of the functions (3) in the interval (x_{j-1},x_j). (We write $x_0=-\infty$ and $x_N=\infty$ for convenience.) This defines a sequence of N terms

(4) $$i(1),i(2),\cdots,i(N),$$

Received August 26, 1964.

American Journal of Mathematics 87:684–694 (1965)

Harold Davenport (1907–1965)

Andrzej Schinzel (1937–2021)

A combinatorial problem

Consider a sequence over the alphabet $\{1,\ldots,n\}$ such that

- $\ldots\ i\ i\ \ldots$ does not appear
- $\underbrace{\ldots i \ldots j \ldots i \ldots j \ldots}_{s+2 \text{ times}}$ does not appear

How long can such a sequence be?

Davenport-Schinzel sequences

Davenport-Schinzel sequence of order s (over alphabet of size n) is sequence that does not contain the following:

- $\ldots\ i\ i\ \ldots$ no two consecutive symbols are the same
- $\ldots\underbrace{i\ldots j\ldots i\ldots j}_{s+2\text{ times}}\ldots$ no alternating subsequence of length $s+2$

Example $(n = 9,\ s = 2)$

- 6, 4, 5, 6, 1, 2, 2, 7, 3
- 2, 5, 1, 2, 7, 8, 7, 1, 3, 4
- 3, 6, 4, 2, 5, 1, 5, 9, 8, 9, 7

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- 3, 6, 4, 2, 5, 1, 5, 9, 8, 9, 7 V

Exercise: Determine the maximal possible length of a DS-sequence of order s as a function of n, for $s = 1$, $s = 2$, $s = 3$, $\ldots$

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$\mathrm{DS}_s(n)$:= maximum length of DS-sequence of order s on n symbols

- $s = 1$:

- $s = 2$:

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- $s = 1$: $\left.\begin{array}{l}\text{possible sequence: } 1, 2, 3, \ldots, n \\ \text{no symbol can appear twice}\end{array}\right\} \implies \quad \mathrm{DS}_1(n) = n$

- $s = 2$:

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- $s = 2$: possible sequence $1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1$

  $\Longrightarrow \quad \mathrm{DS}_2(n) \geqslant 2n - 1$

  Proof by induction, remove symbol whose first occurrence is last, plus at most one adjacent symbol:

  $\mathrm{DS}_2(n) \leqslant \mathrm{DS}(n-1) + 2 \Longrightarrow \quad \mathrm{DS}_2(n) \leqslant 2n - 1$

Theorem. $\mathrm{DS}_s(n)$ is near-linear for any constant s. In particular,

- $\mathrm{DS}_1(n) = n$
- $\mathrm{DS}_2(n) = 2n - 1$
- $\mathrm{DS}_3(n) = \Theta(n\alpha(n))$
- $\mathrm{DS}_s(n) = o(n \log^* n)$ for any fixed constant s

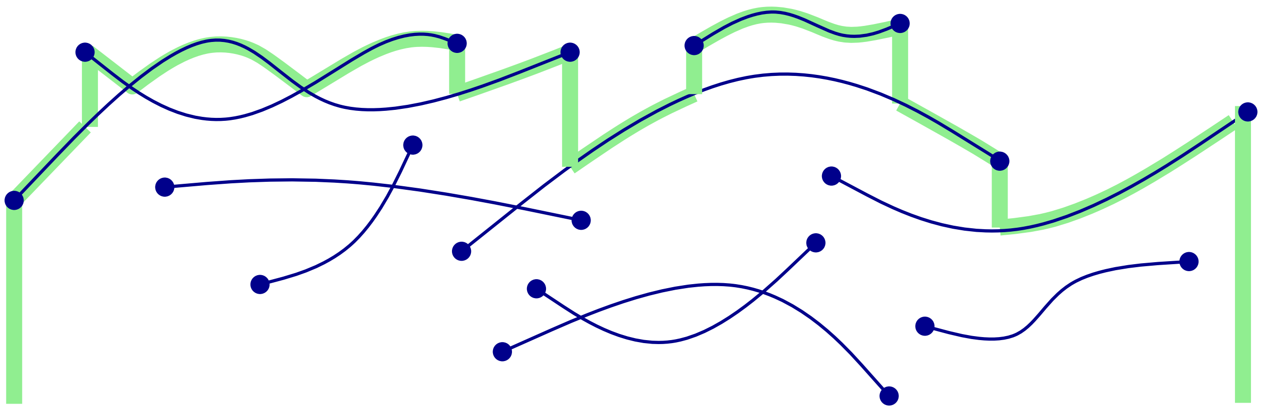
where $\alpha(n)$ is the inverse Ackermann function

$\alpha(n)$ grows slower than super-super-super-super-super-slowly ...

$\alpha(n)$ is inverse of Ackermann function $A(n)$, where $A(n) = A_n(n)$ with:

$$\begin{array}{rcll} A_1(n) & = & 2n & \text{for } n \geqslant 1 \\ A_k(1) & = & 2 & \text{for } k \geqslant 1 \\ A_k(n) & = & A_{k-1}(A_k(n-1)) & \text{for } k \geqslant 2 \text{ and } n \geqslant 2 \end{array}$$

$A(1) = 2$, $A(2) = 4$, $A(3) = 16$, $A(4) =$ tower of 65536 2's

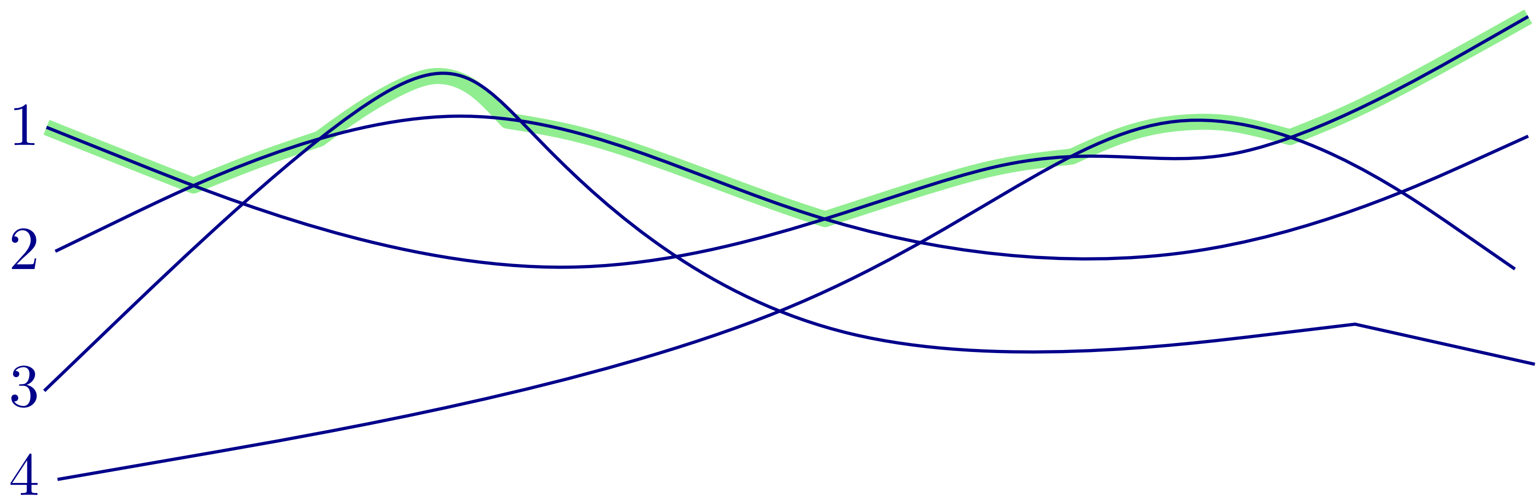


back to upper envelopes

Theorem. Let S be a set of n **infinite** x-monotone curves such that any two curves intersect at most s times. Then the maximum complexity of the upper envelope of S is $O(\mathrm{DS}_s(n))$.

for example: $O(n)$ for lines

Proof.



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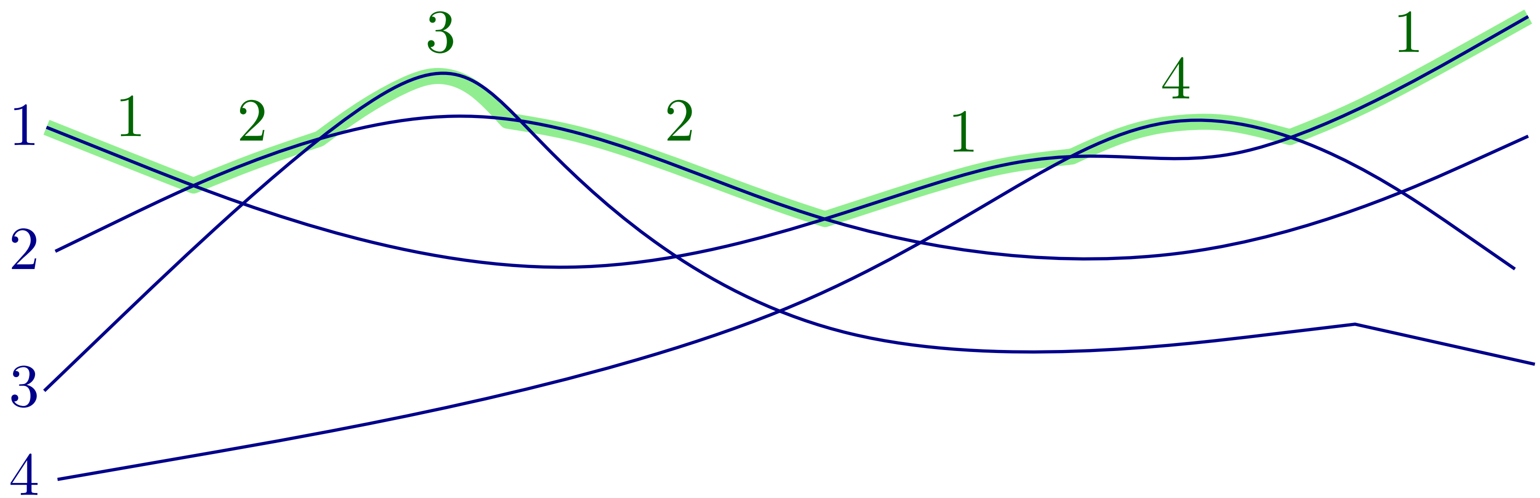

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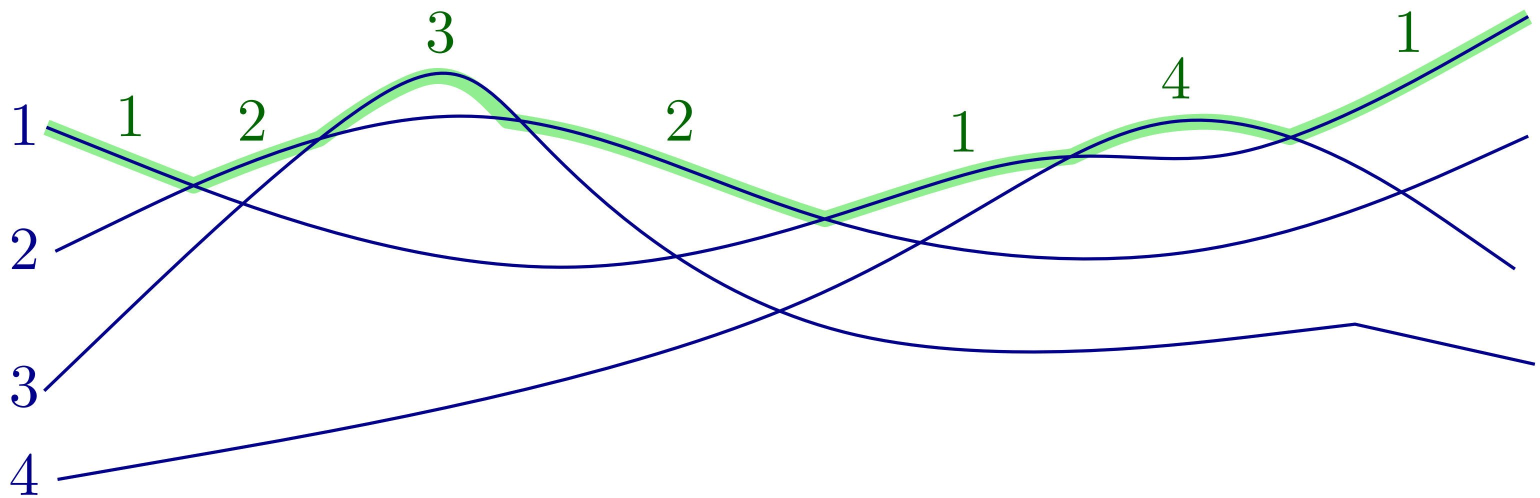
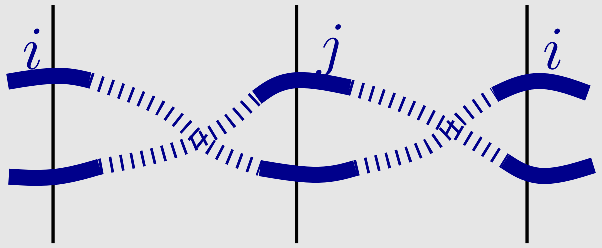


alternating sequence of length t implies $t - 1$ intersections



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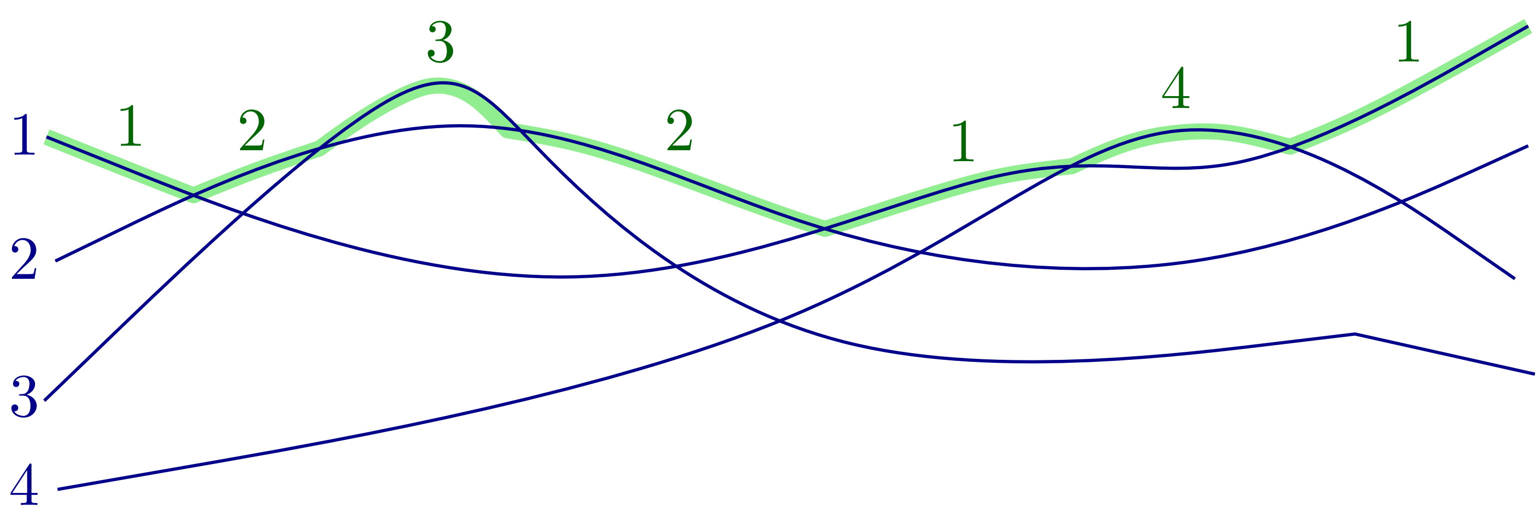
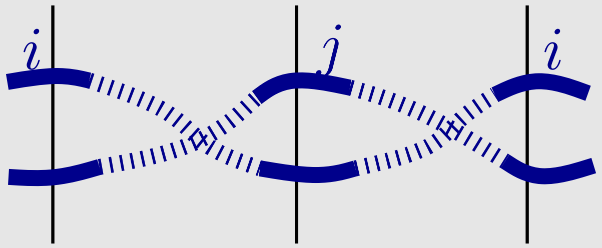


we cannot have alternating sequence of length $s+2$

$\Longrightarrow$ DS(n, s)-sequence

alternating sequence of length t implies $t-1$ intersections



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The Complexity of Upper Envelopes

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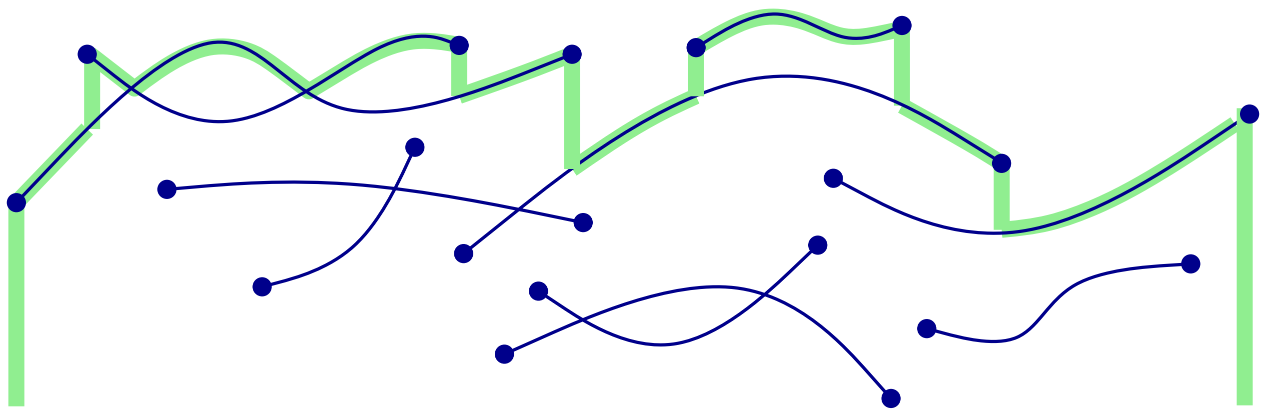

The Complexity of Upper Envelopes

Theorem. Let S be a set of n ~~infinite~~ x-monotone curves such that any two curves intersect at most s times. Then the maximum complexity of the upper envelope of S is $O(\mathrm{DS}_{s+2}(n))$.

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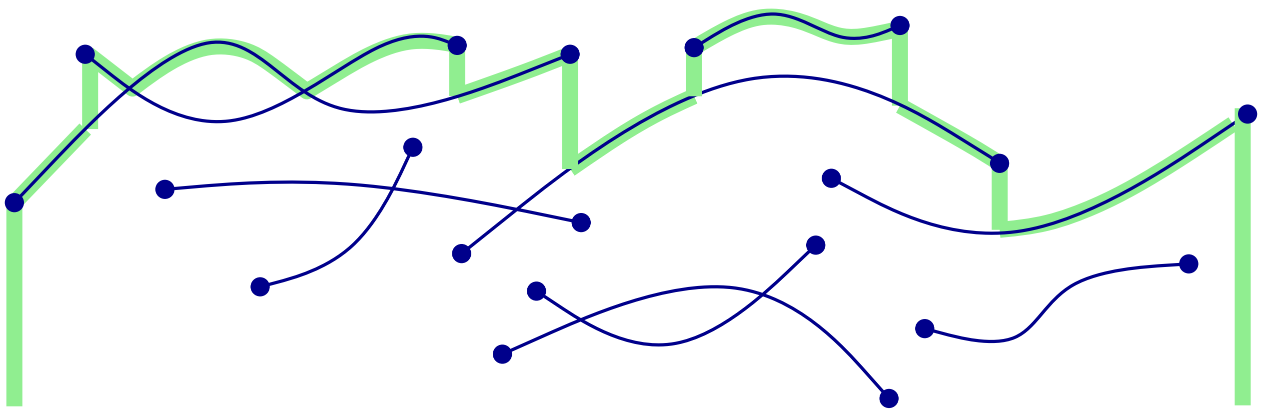
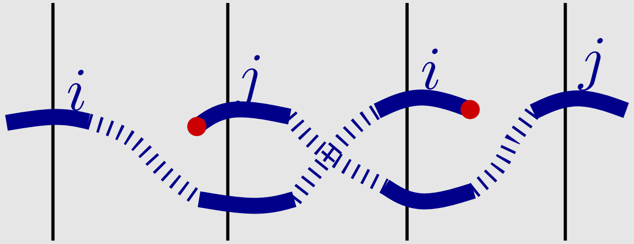


alternating sequence of length t implies $t-3$ intersections



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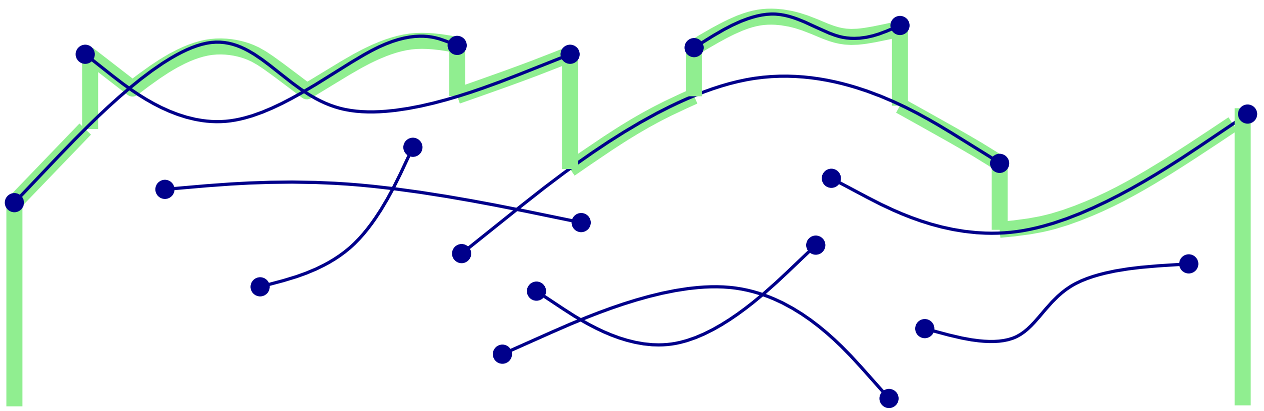
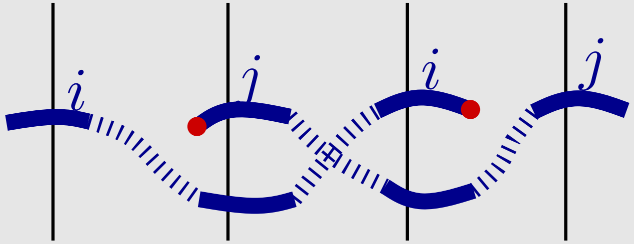


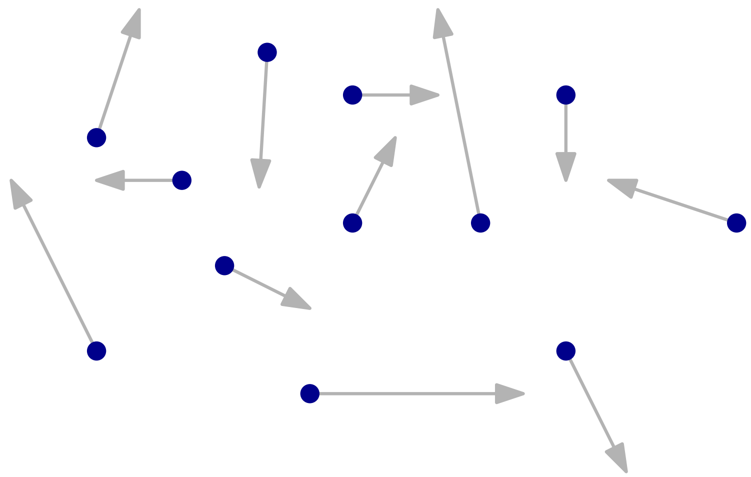
we cannot have alternating sequence of length $s+4$
$\Longrightarrow$ DS($n, s+2$)-sequence

alternating sequence of length t implies $t-3$ intersections



■

P: set of n points in $\mathbb{R}^2$ that move linearly (or: on polynomial trajectories)



- How often can the closest pair change, in the worst case?
- How often can the convex hull change, in the worst case?
- How often can the Delaunay triangulation change, in the worst case?

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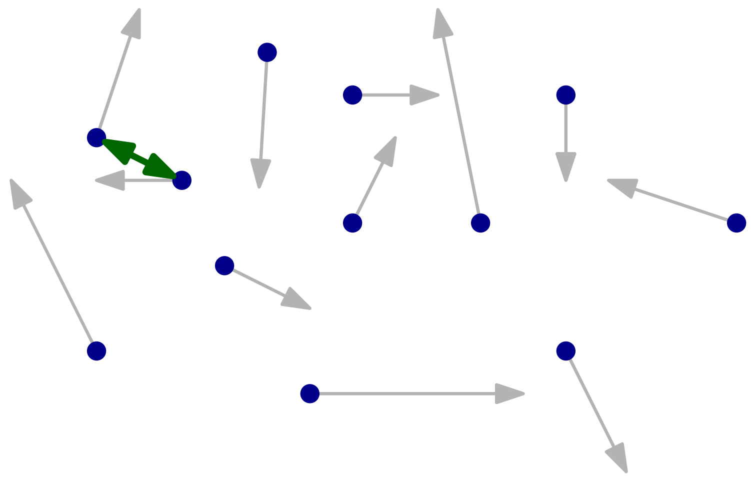

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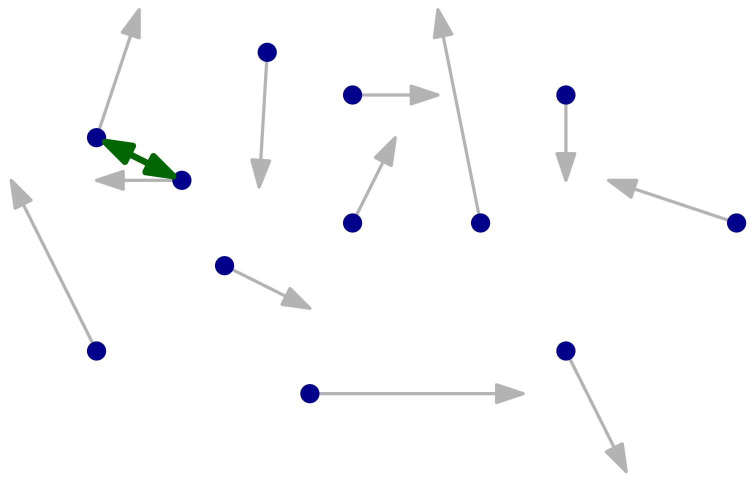

Lower bound

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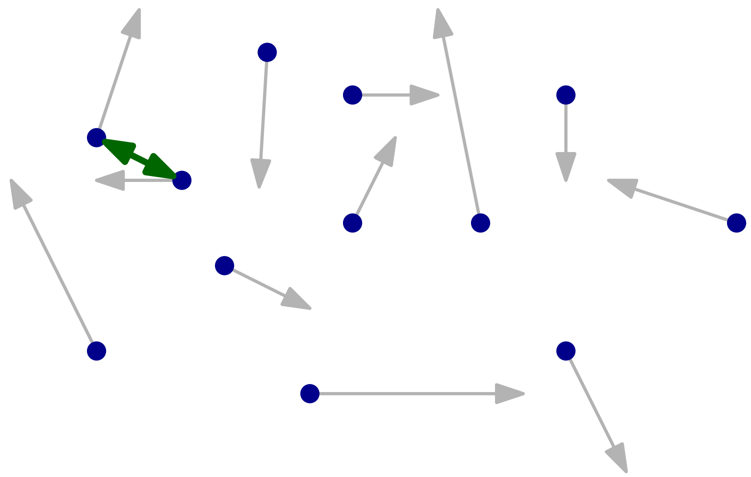

Lower bound

$\Omega(n^2)$ changes

Upper Envelopes: Applications for Moving Ponits

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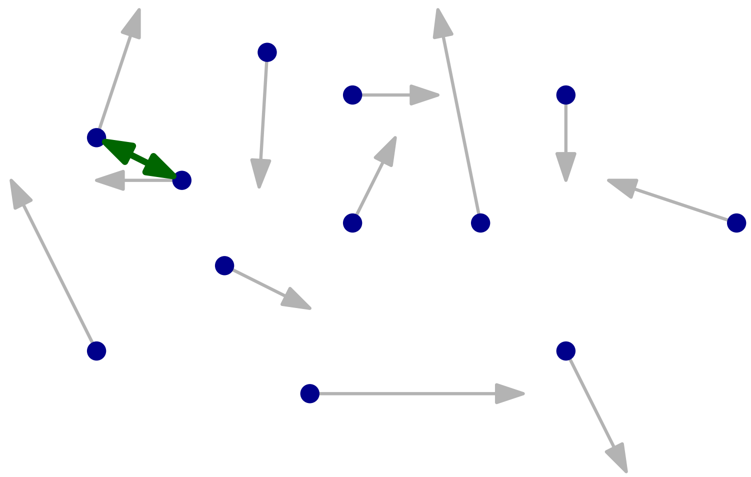

Upper bound

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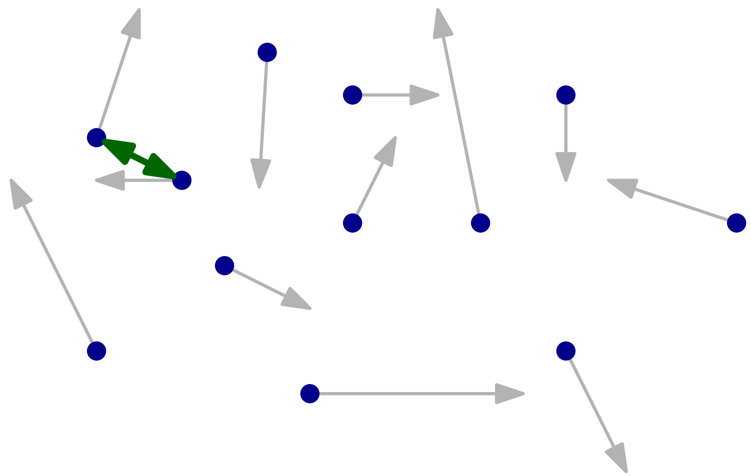
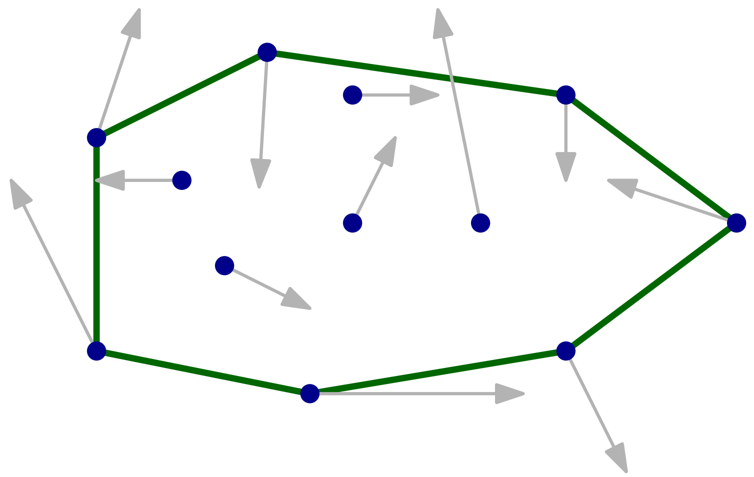


Upper bound

- for each pair p, q define $f_{pq}(t) :=$ distance between p and q at time t
- number of changes = complexity of lower envelope of n^2 functions
  $\approx O(n^2)$

How often can the convex hull change, in the worst case?



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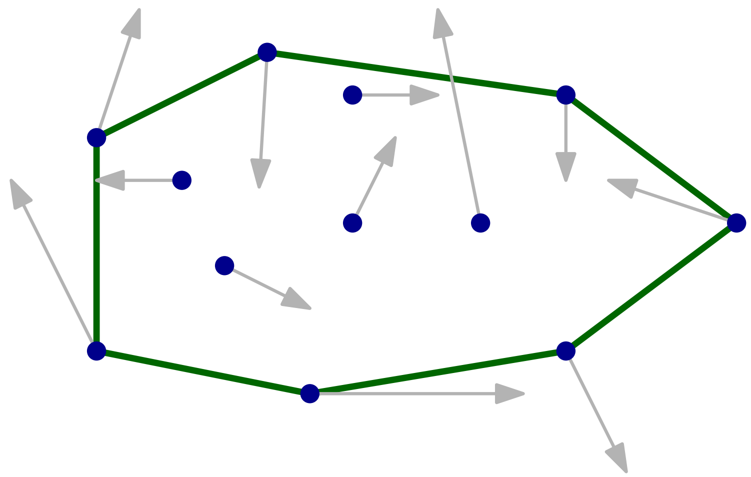

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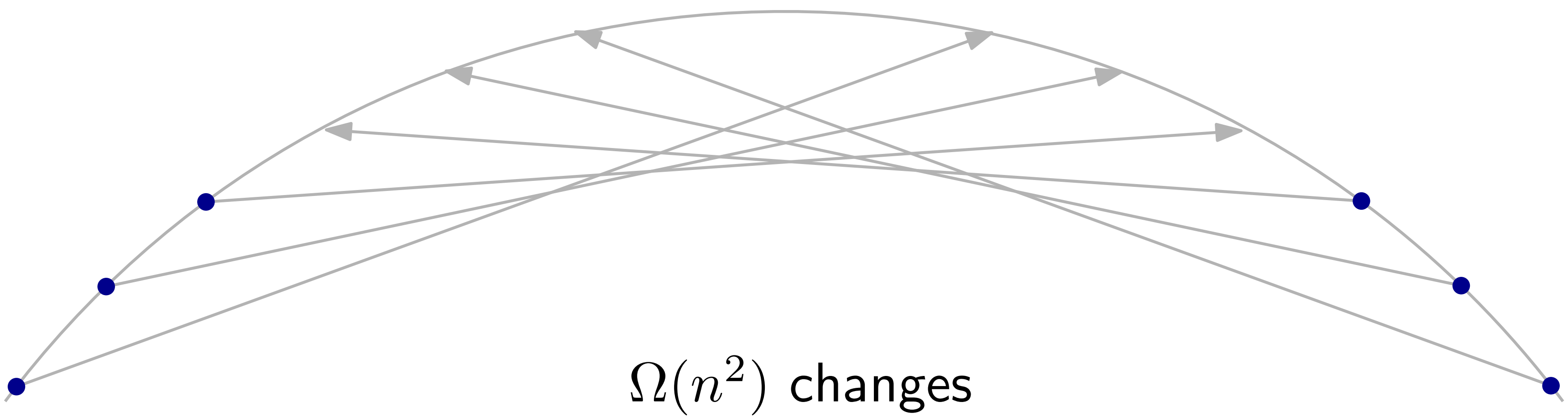

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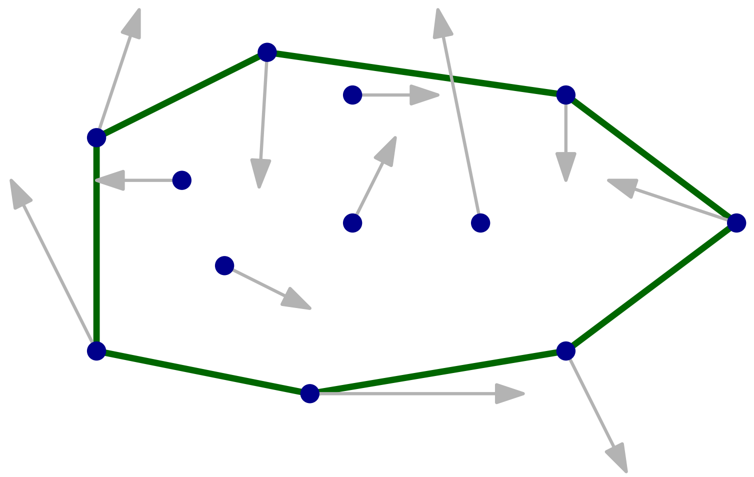

Trivial upper bound

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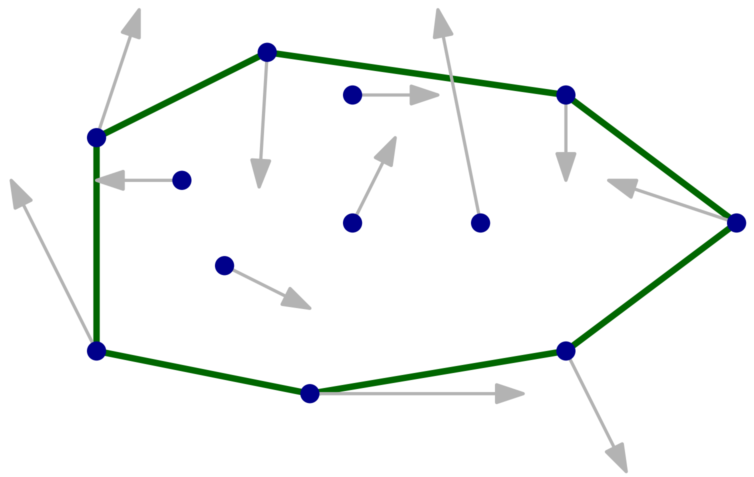


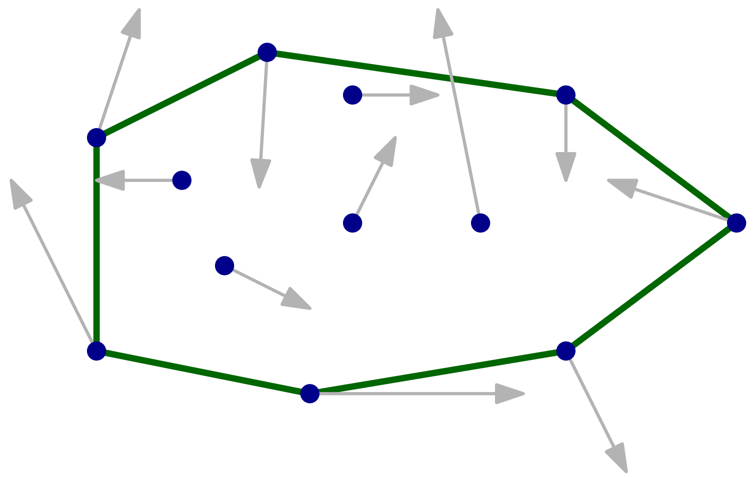
Trivial upper bound

convex hull changes $\Longrightarrow$ three points become collinear

$\Longrightarrow$ happens $O(1)$ times for each triple

$\Longrightarrow$ $O(n^3)$ changes to convex hull

How often can the convex hull change, in the worst case?



A better bound using upper envelopes

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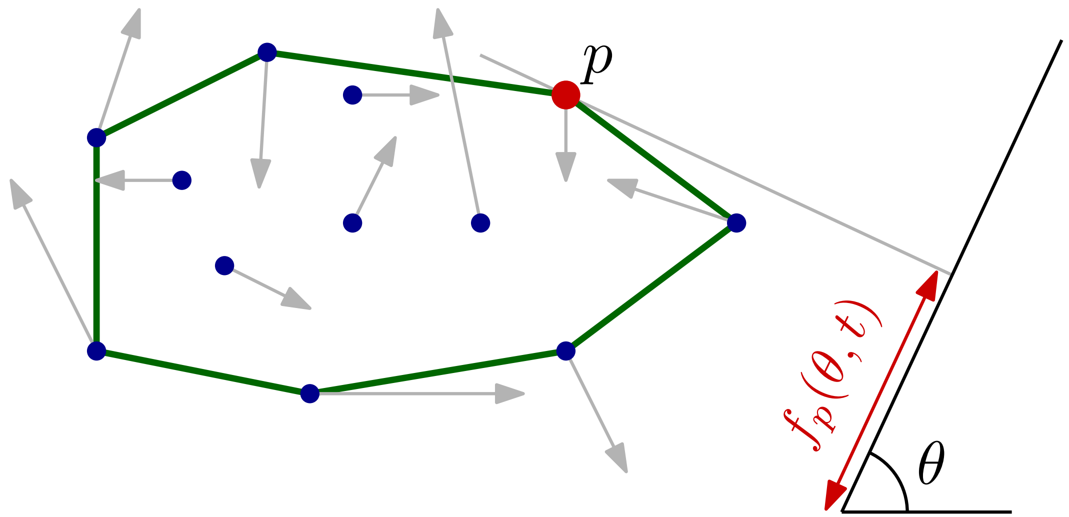

A better bound using upper envelopes

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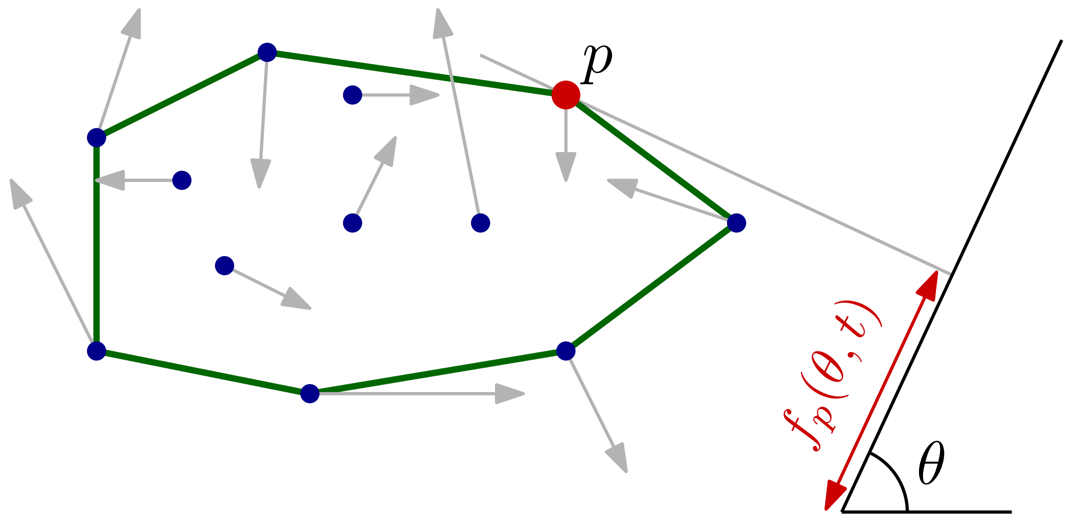

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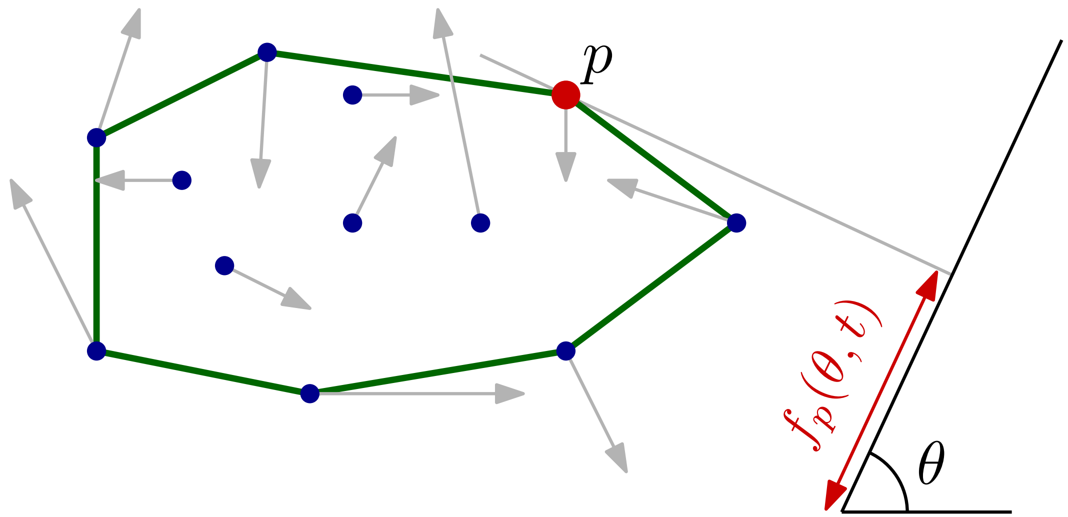

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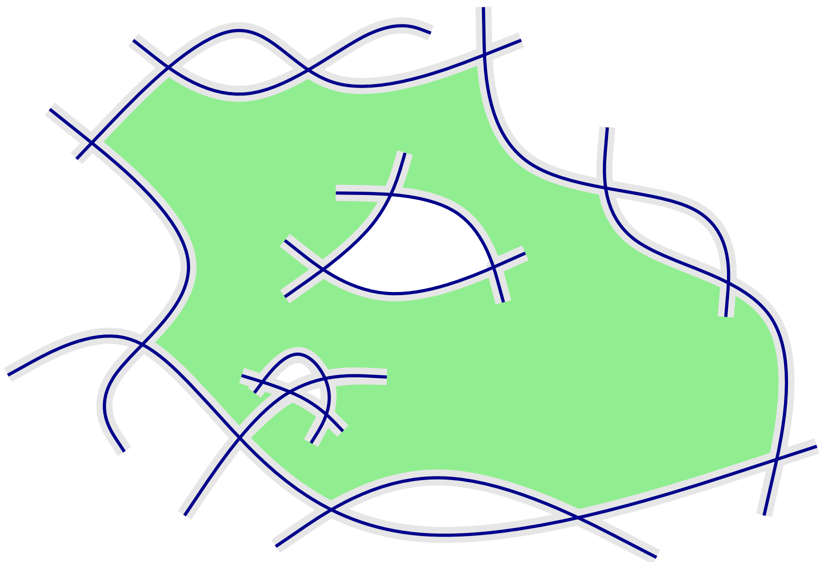
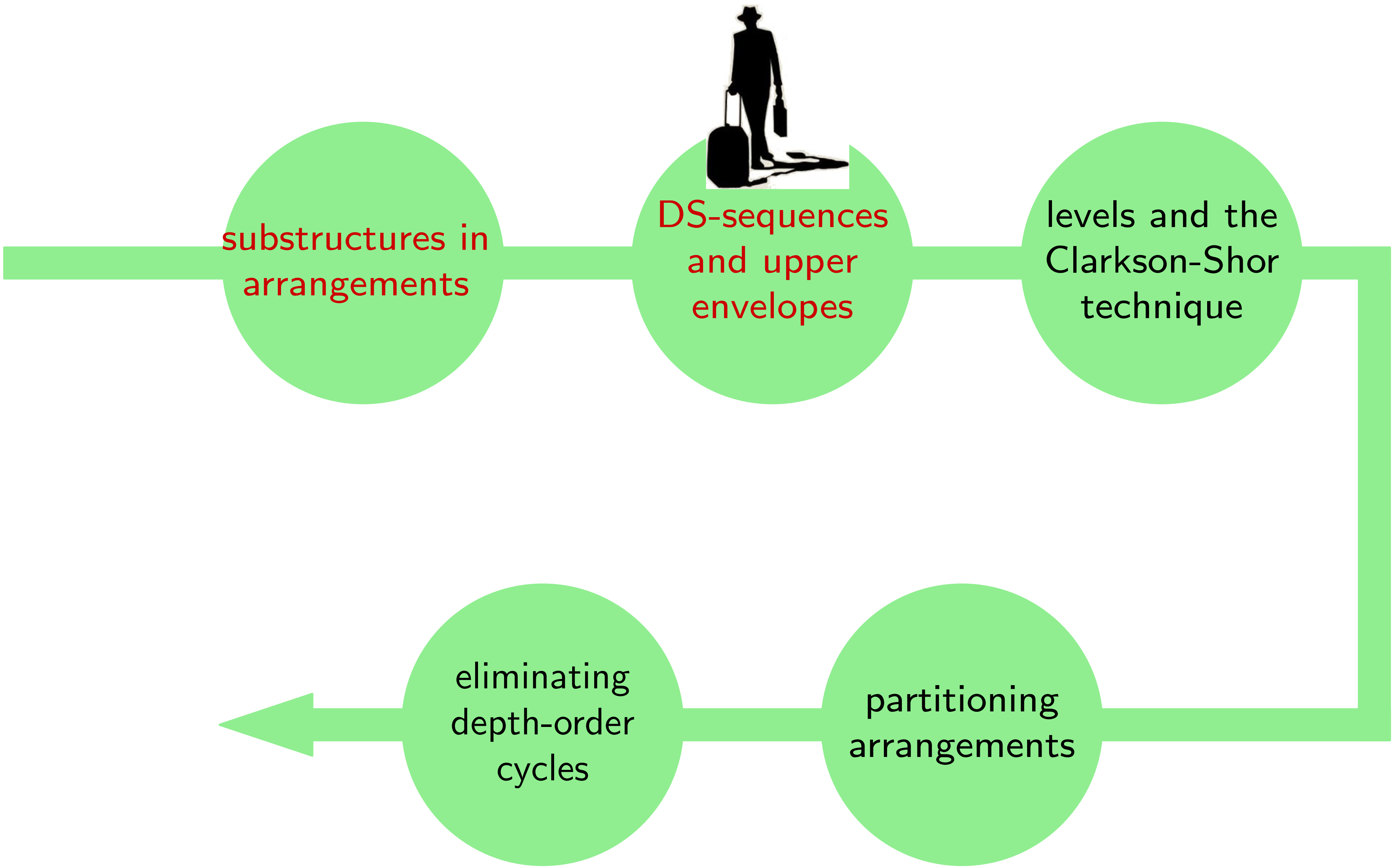
- number of changes $= O($complexity of upper envelope of surfaces in $\mathbb{R}^3) = O(n^{2+\varepsilon})$

Theorem. Let S be a set of n curves in the plane such that any two curves intersect at most s times. Then the maximum complexity of a single cell of $\mathcal{A}(S)$ is $O(\mathrm{DS}_{s+2}(n))$.

for example: $O(n\alpha(n))$ for line segments



proof also uses Davenport-Schinzel sequences but is more complicated


substructures in arrangements
DS-sequences and upper envelopes
levels and the Clarkson-Shor technique
partitioning arrangements
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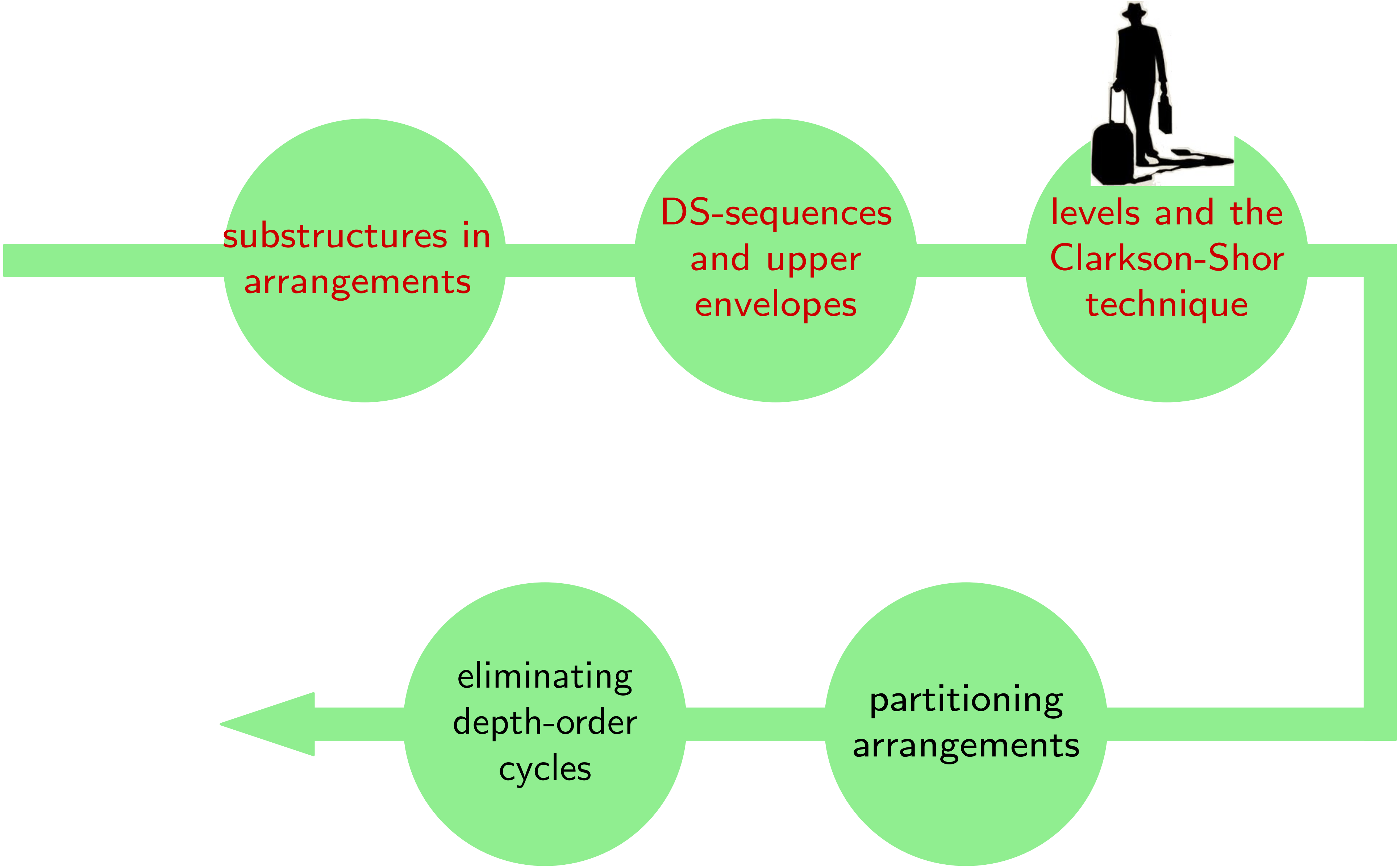
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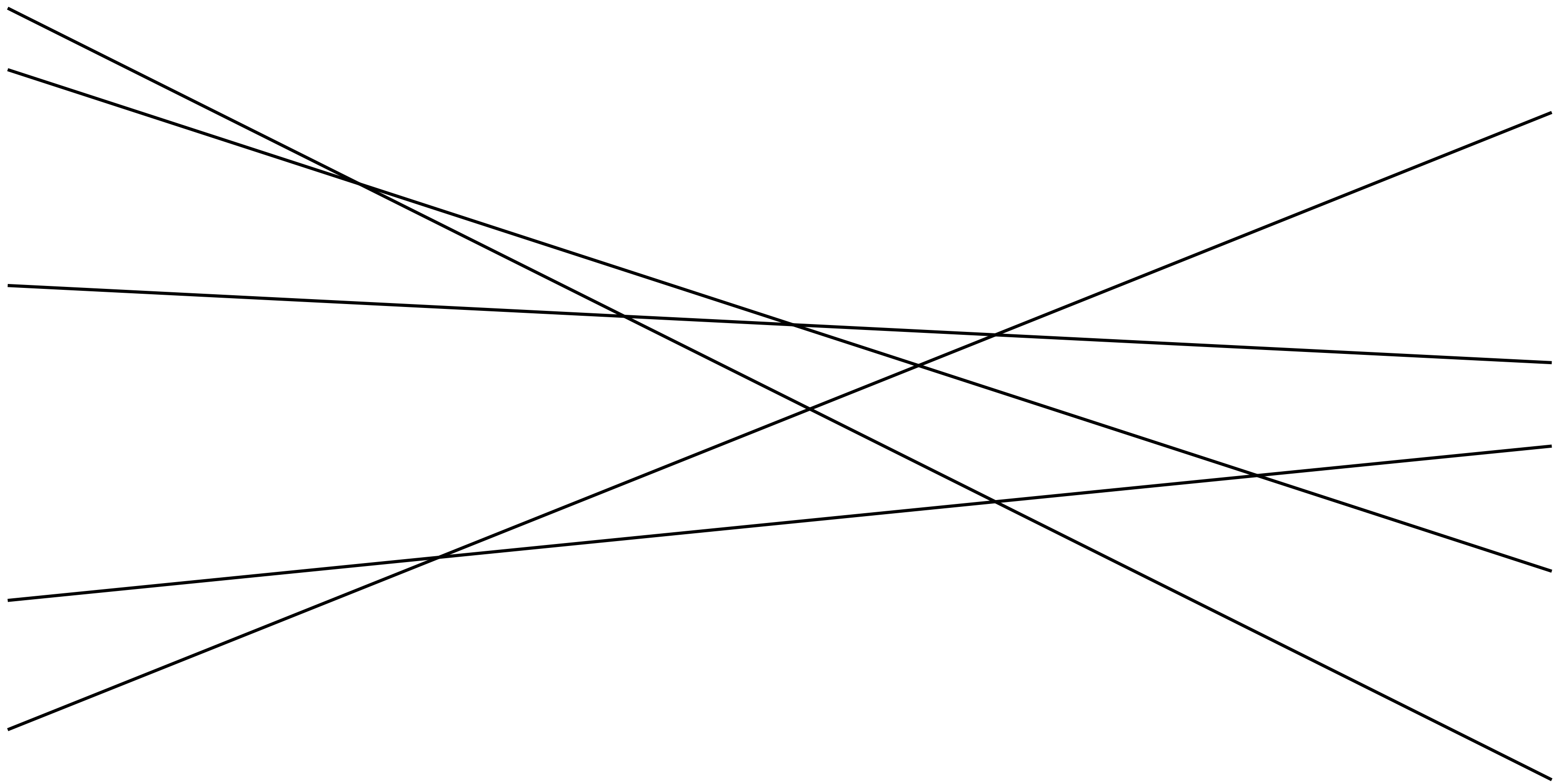
Levels in arrangements



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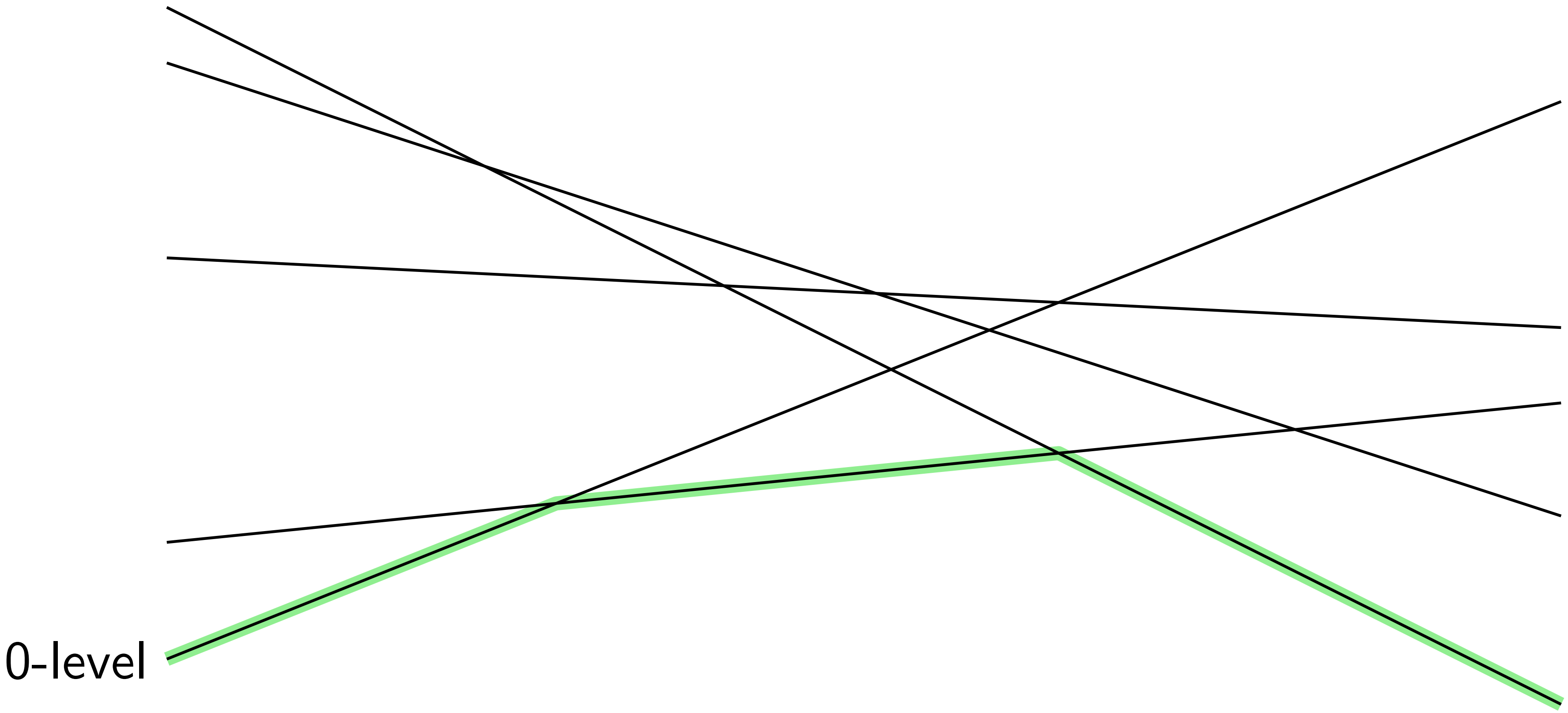

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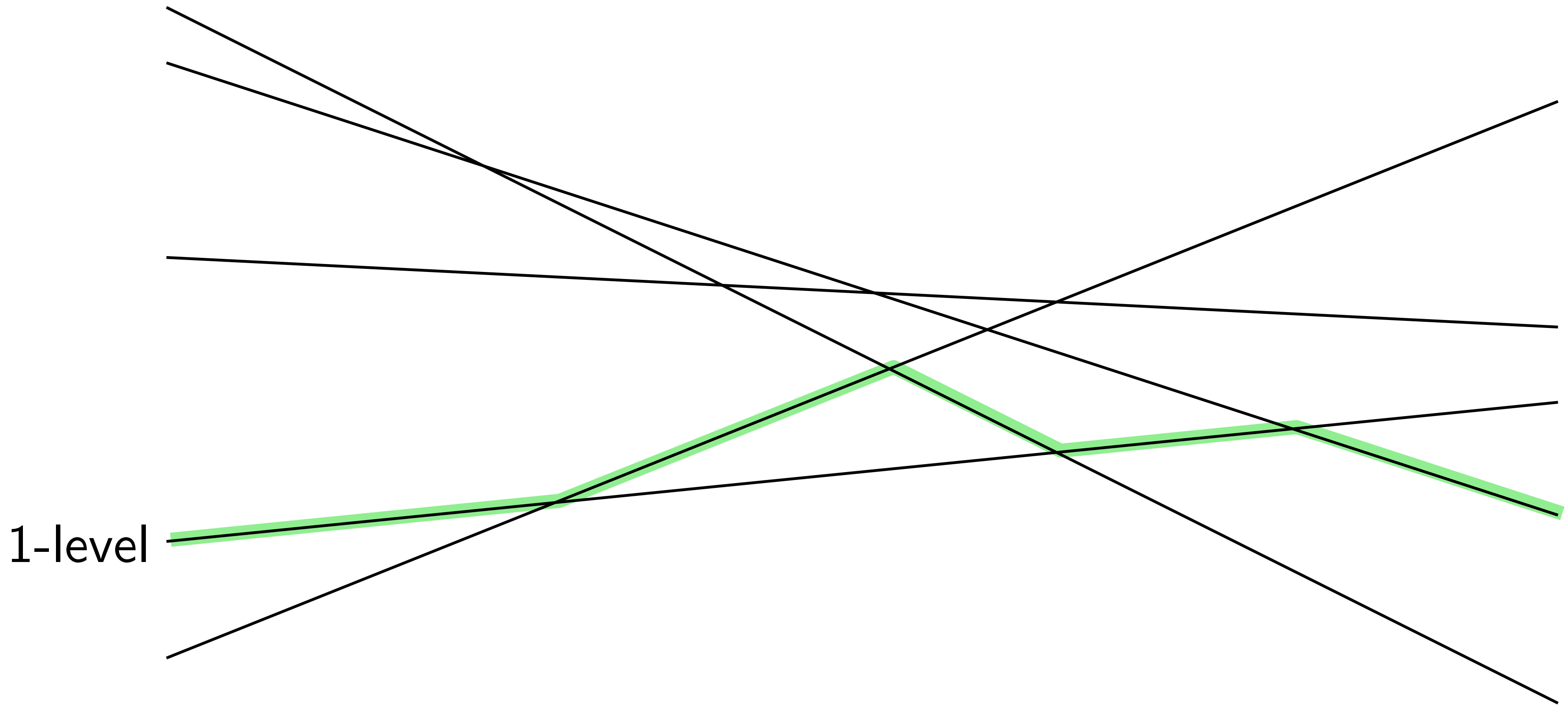
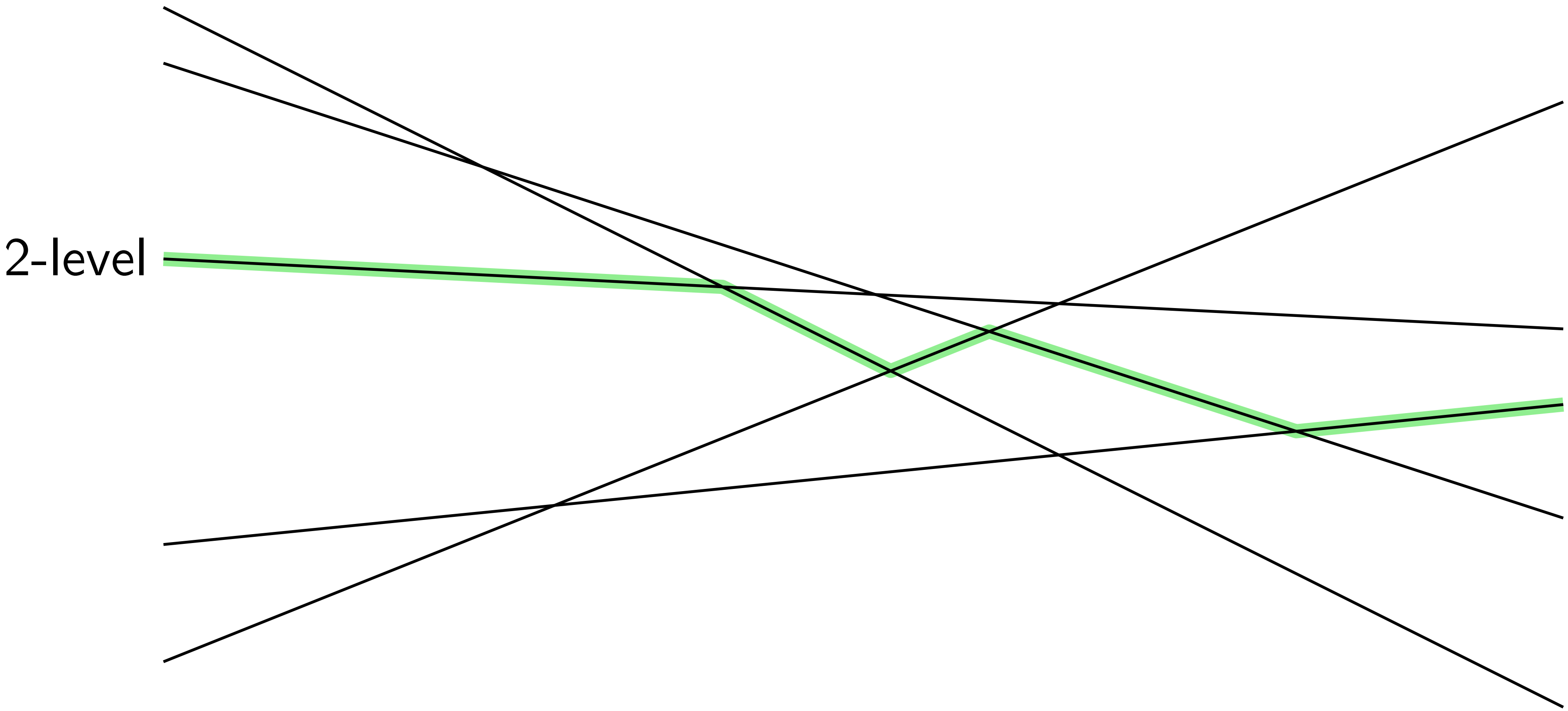
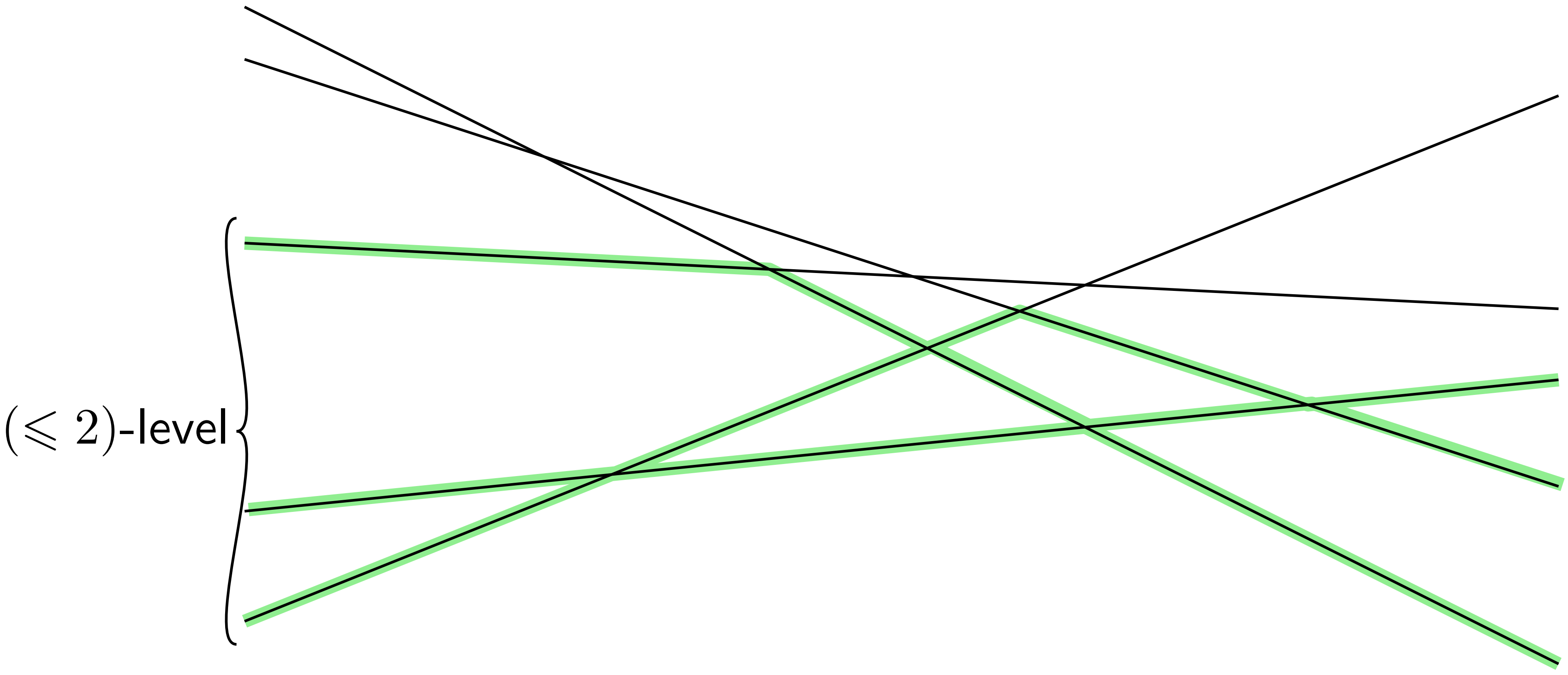
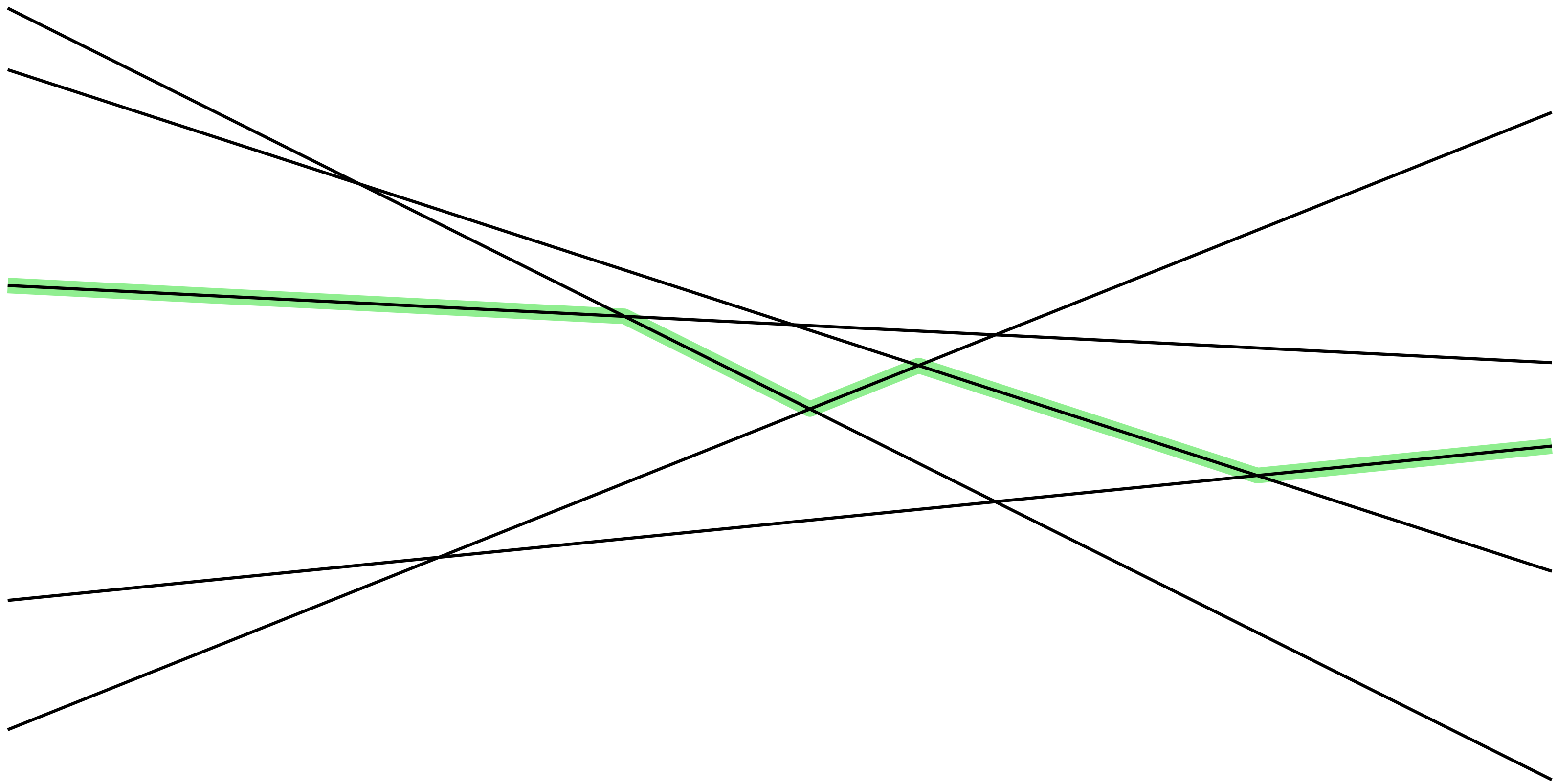



2-level

Levels in arrangements

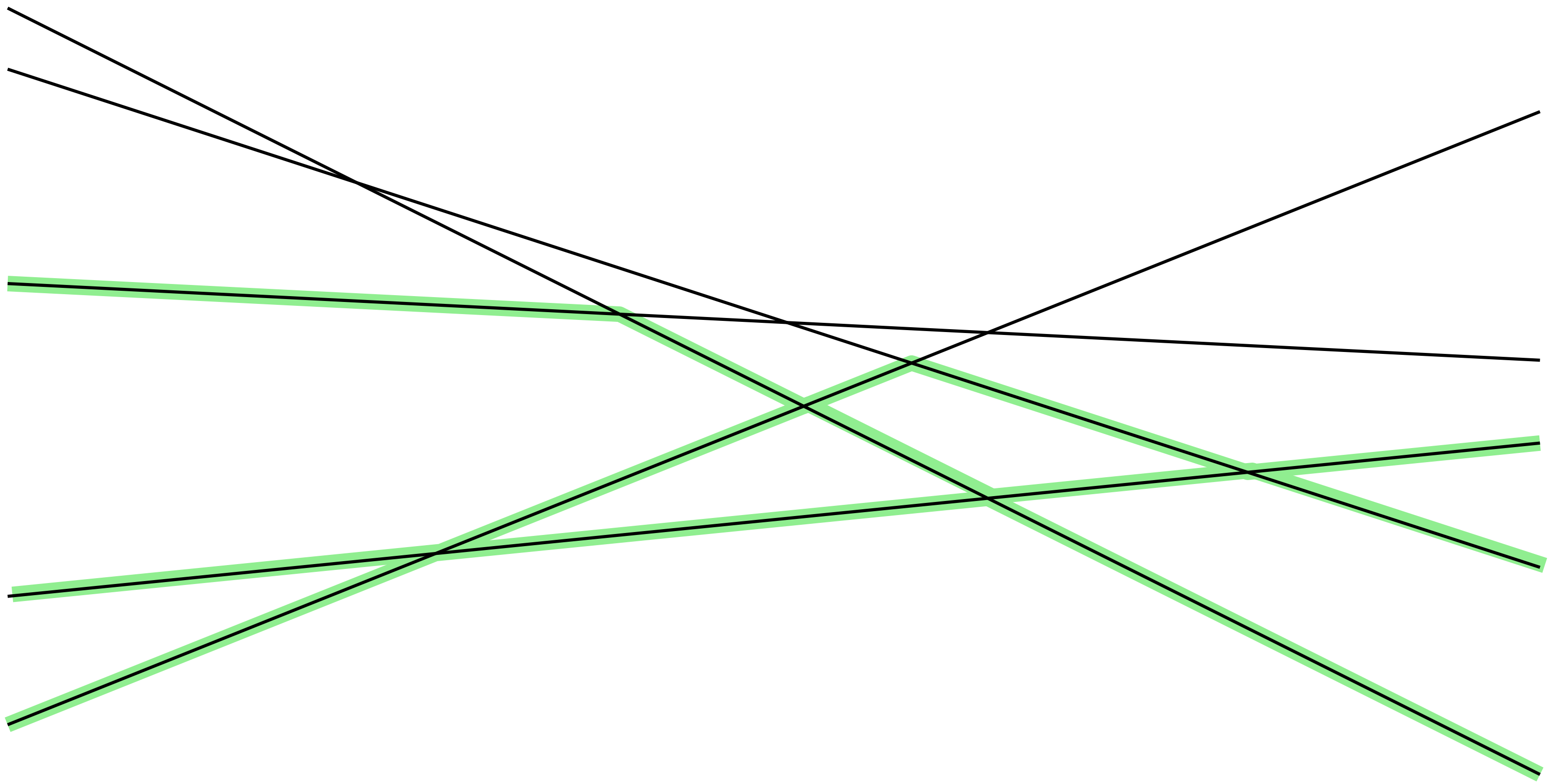


Levels in arrangements

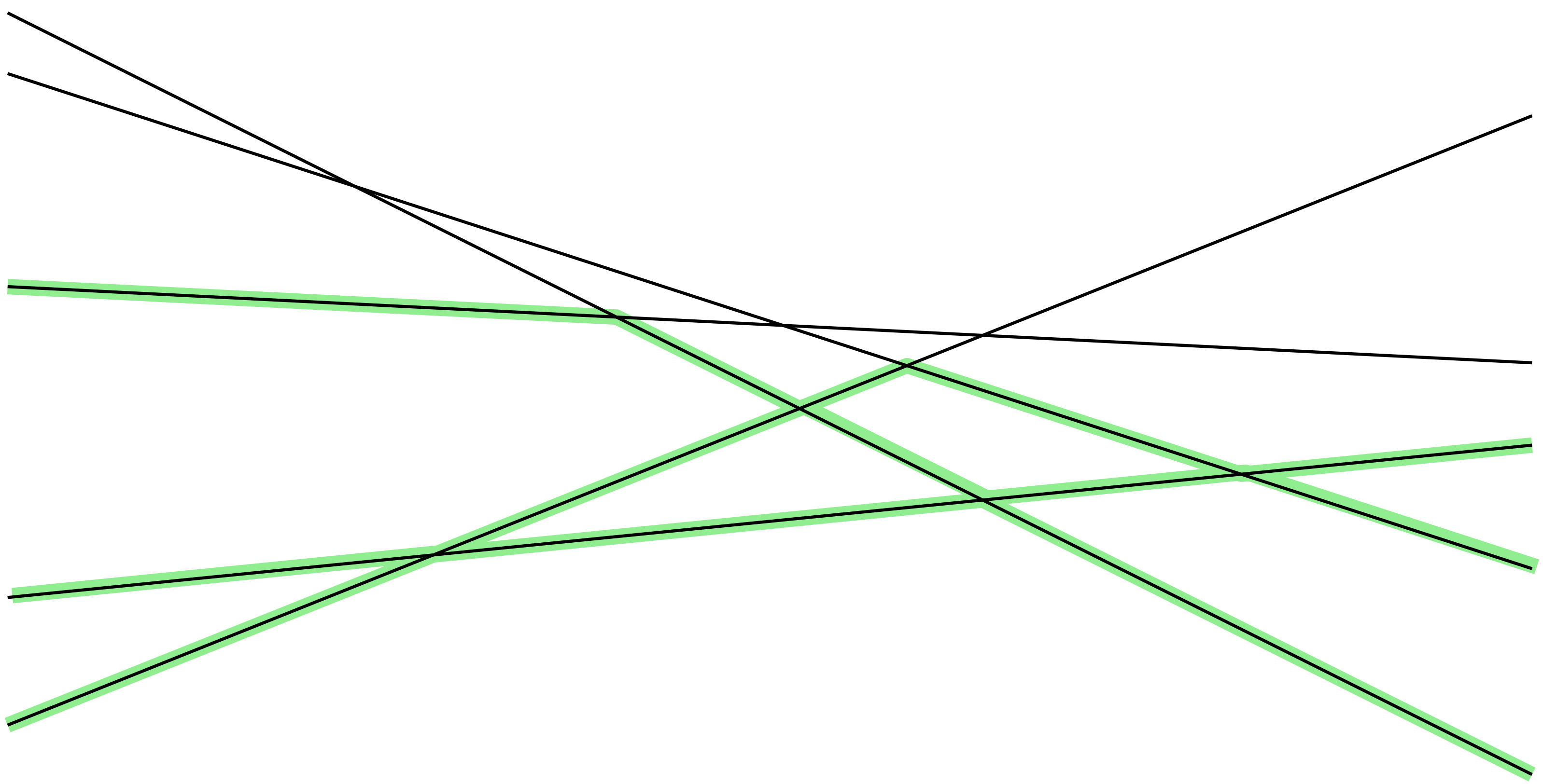


What is the max complexity of the k-level in an arrangement of n lines?

- 0-level $=$ lower envelope $\implies$ complexity $\leqslant n$
- $k \geqslant 1$: complexity is $n2^{\Omega(\sqrt{\log k})}$ and $O(nk^{1/3})$ major open problem



What is the max complexity of the $(\leqslant k)$-level in an arrangement of n lines?



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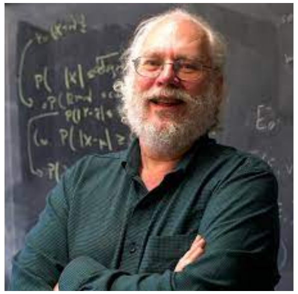

Clarkson-Shor '89: $\Theta(nk)$

- in $\mathbb{R}^d$: $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$
- bound for $d = 2$ was already known

Theorem. The max complexity of the $(\leqslant k)$-level in an arrangement induced by a set L of n lines in the plane is $O(nk)$.

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Take sample $R \subset L$ by picking each line $\ell \in L$ with probability $1/k$.

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$$\mathbb{E}\big[\text{complexity of 0-level of } R\big] \;\leqslant\; \mathbb{E}\big[|R|\big] \;=\; n/k$$

The Clarkson-Shor Technique: Application to $(\leqslant k)$-levels

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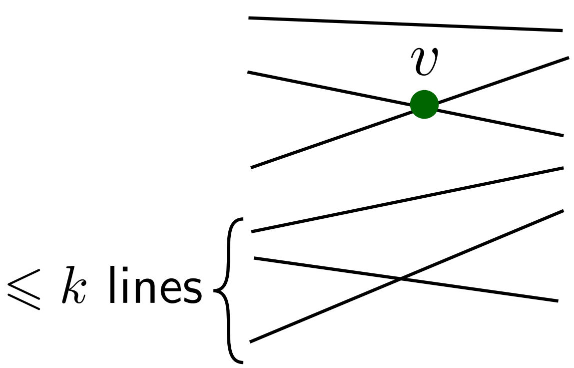

vertex of k-level of L shows up on 0-level of R iff

- both lines defining v are in R
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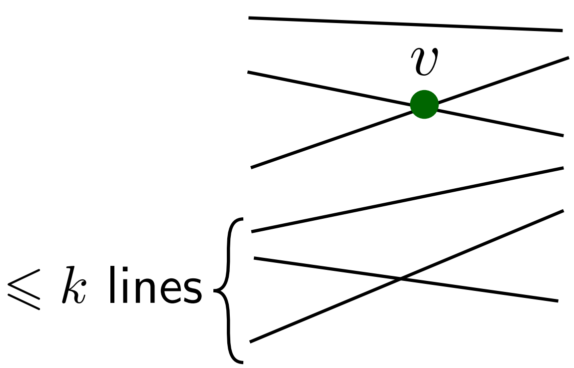

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$$\text{prob} \geqslant \left(\tfrac{1}{k}\right)^2 \cdot \left(1 - \tfrac{1}{k}\right)^k \geqslant \left(\tfrac{1}{k}\right)^2 \cdot \tfrac{1}{e}$$

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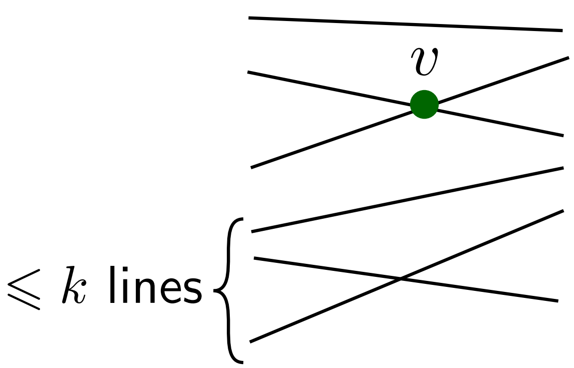

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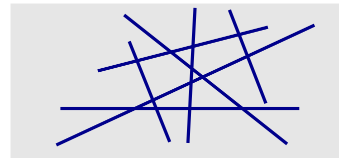
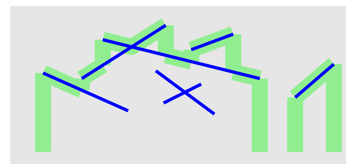
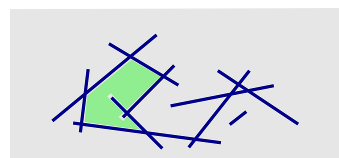
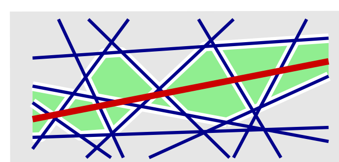
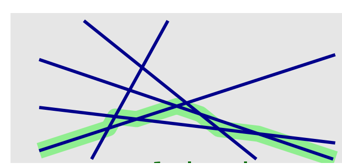
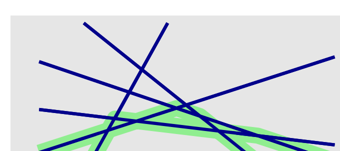
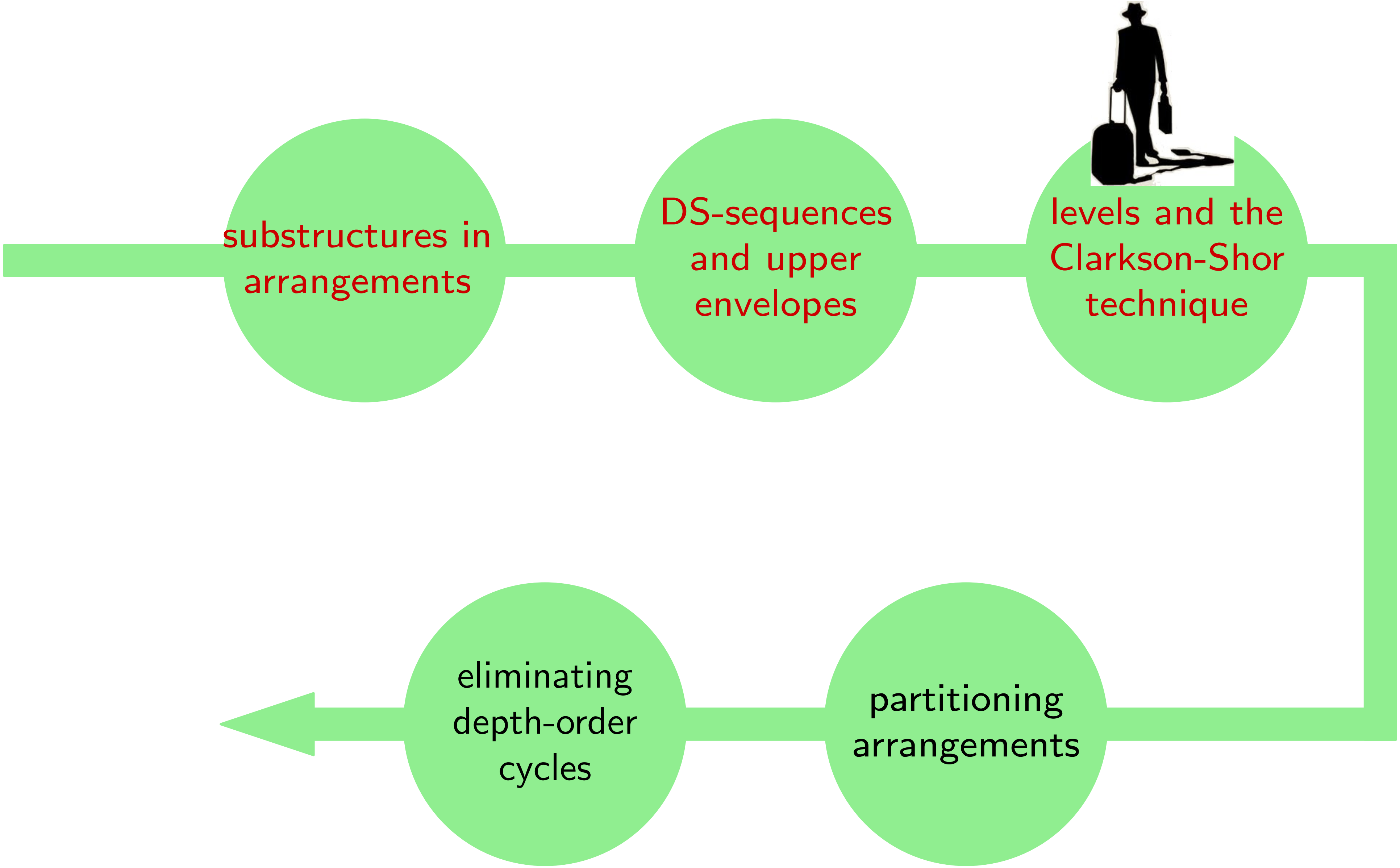
Overview of Complexity of Substructures in Arrangements in $\mathbb{R}^2$

full arrangement		$\Theta(n^2)$
upper envelope (segments)		$\Theta(n\alpha(n))$
single cell (segments)		$\Theta(n\alpha(n))$
zone (lines)		$\Theta(n)$
k-level (lines)		$O(nk^{1/3})$ $\quad n2^{\Omega(\sqrt{\log k})}$
($\leqslant k$)-level (lines)		$\Theta(nk)$

Lecture Overview



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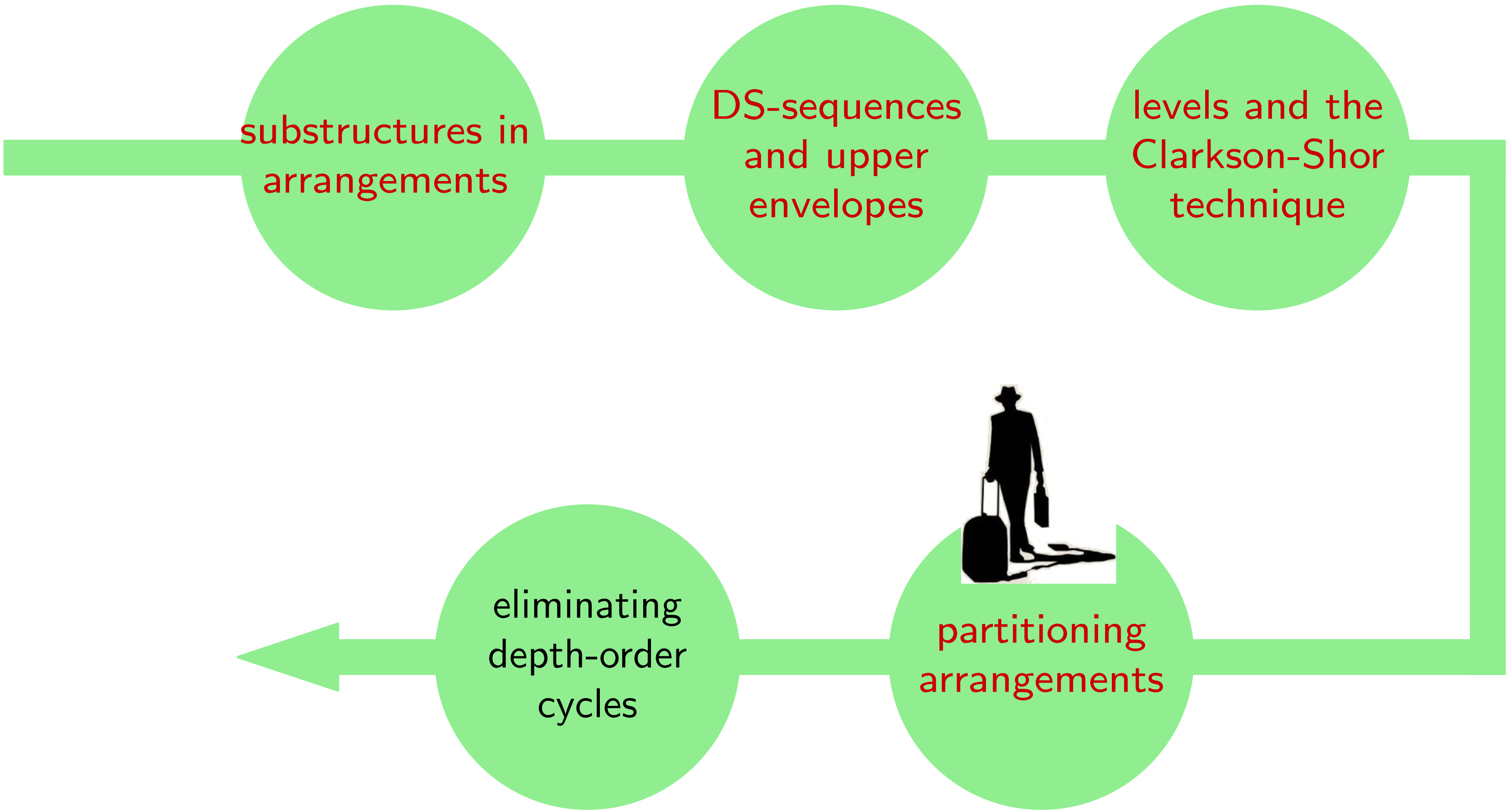

- divide-and-conquer: important algorithmic design technique
- for geometric problems: perform divide step by partitioning space

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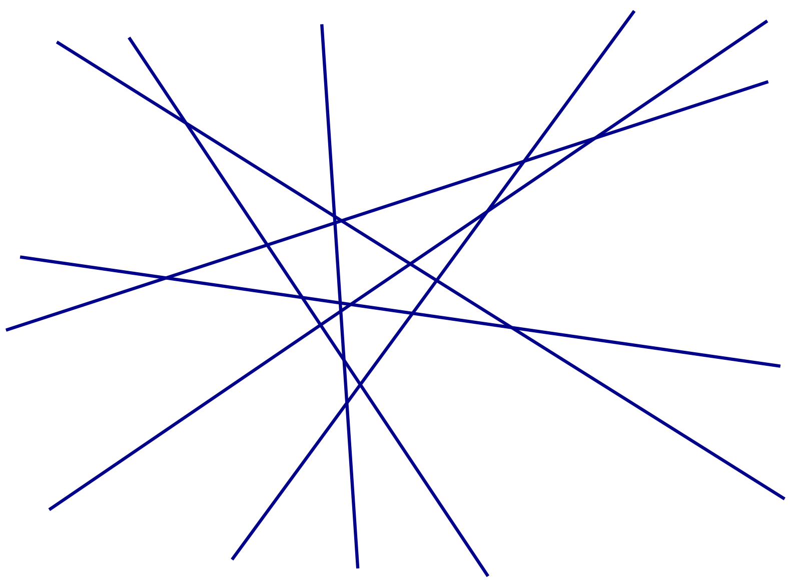
Example: point location in arrangements



Store lines in data structure such that we can find the cell containing a query point q in $O(\log n)$ time

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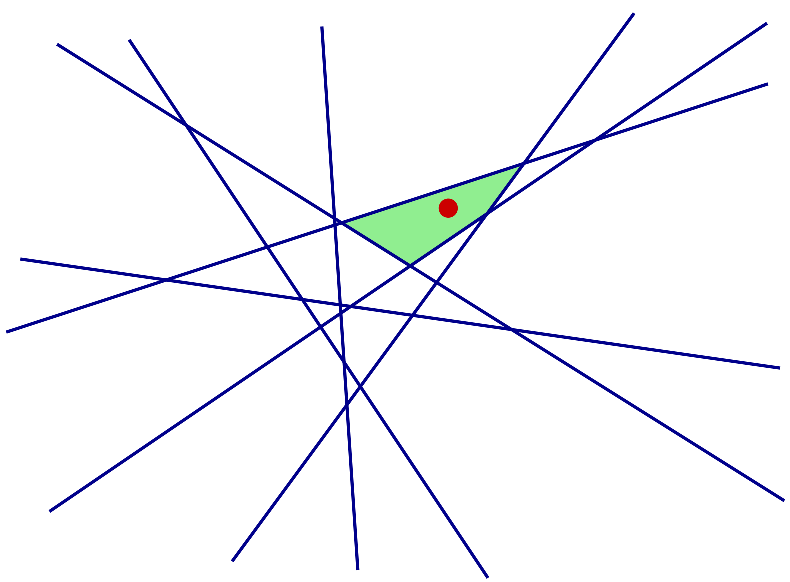

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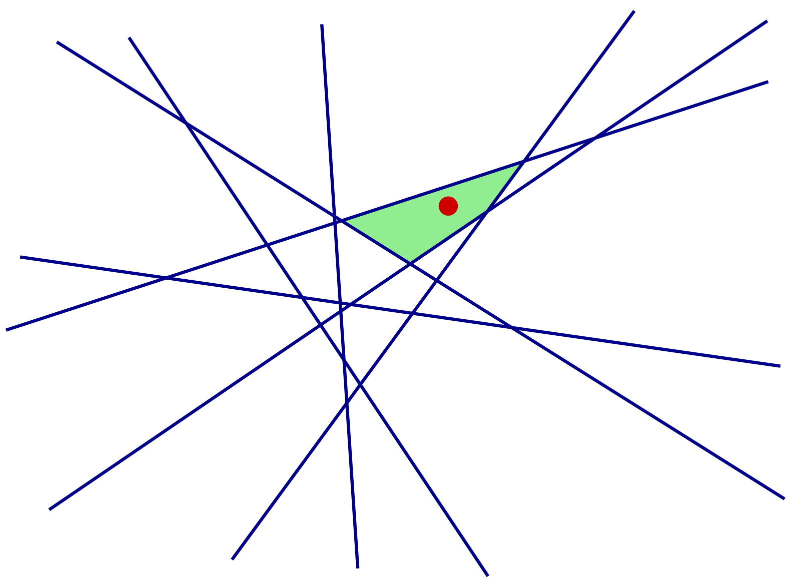

Store lines in data structure such that we can find the cell containing a query point q in $O(\log n)$ time

Idea

- Partition plane into small number of regions
- Find region containing query point q
- Recursively find cell containing q within region

Point Location in Arrangements

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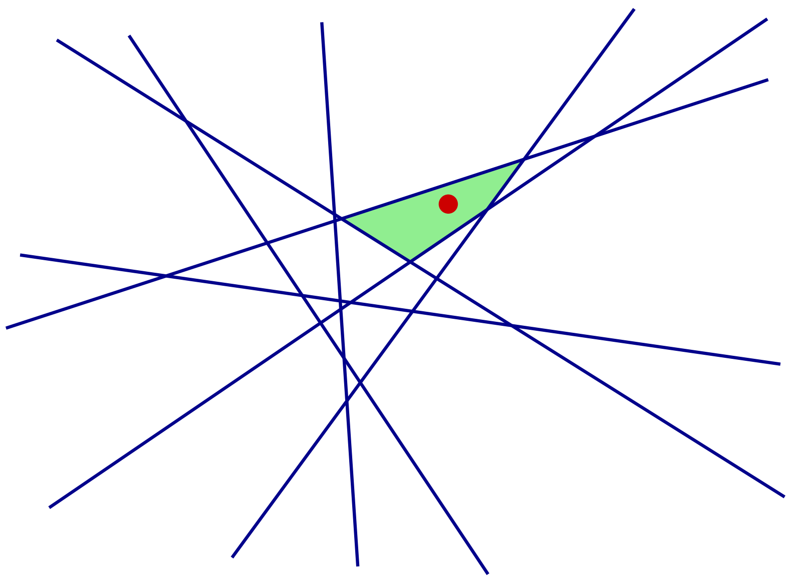
- for geometric problems: perform divide step by partitioning space

Example: point location in arrangements



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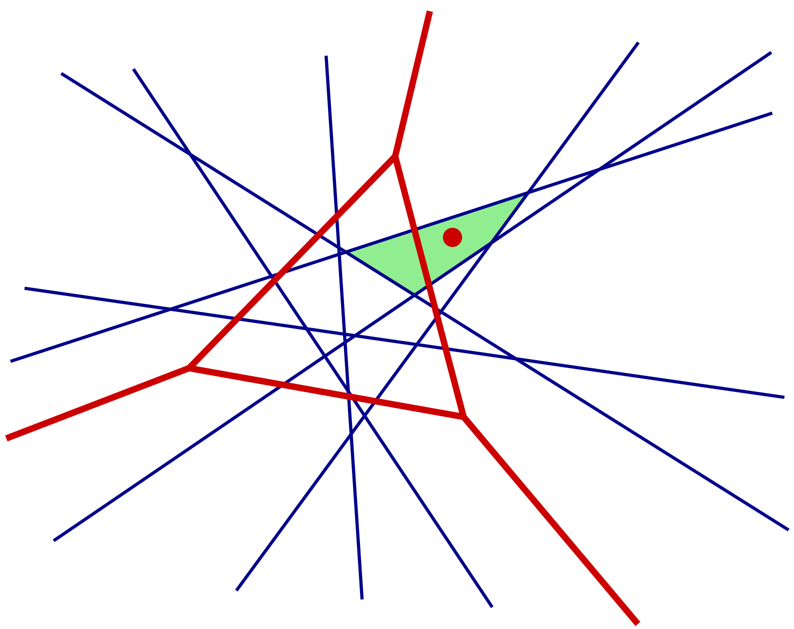

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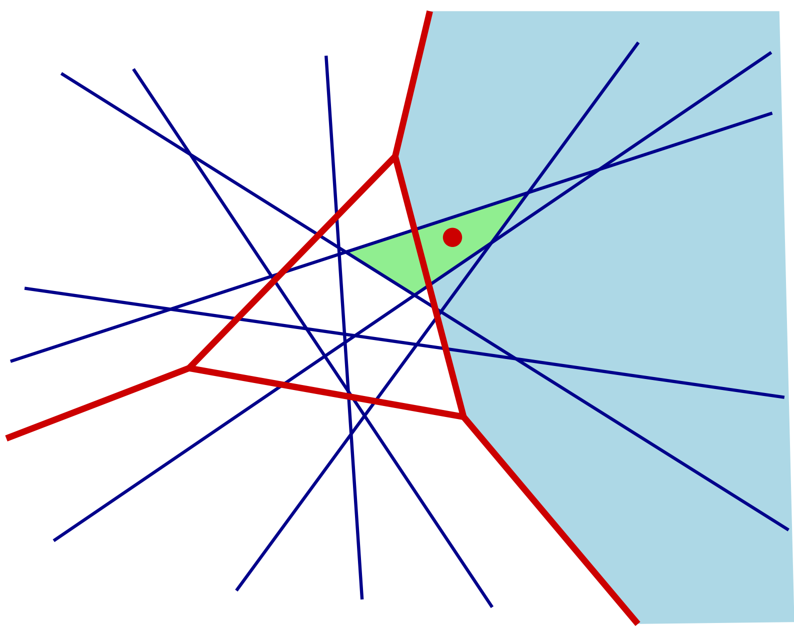
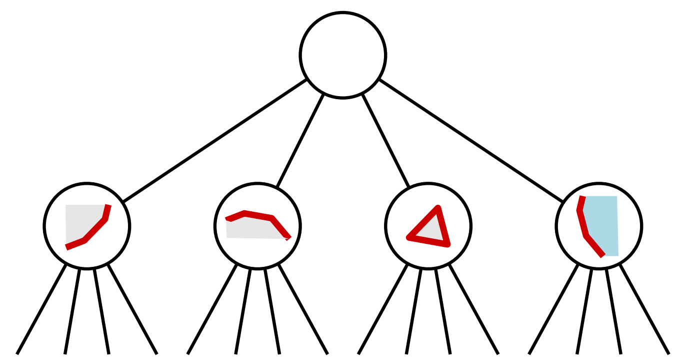
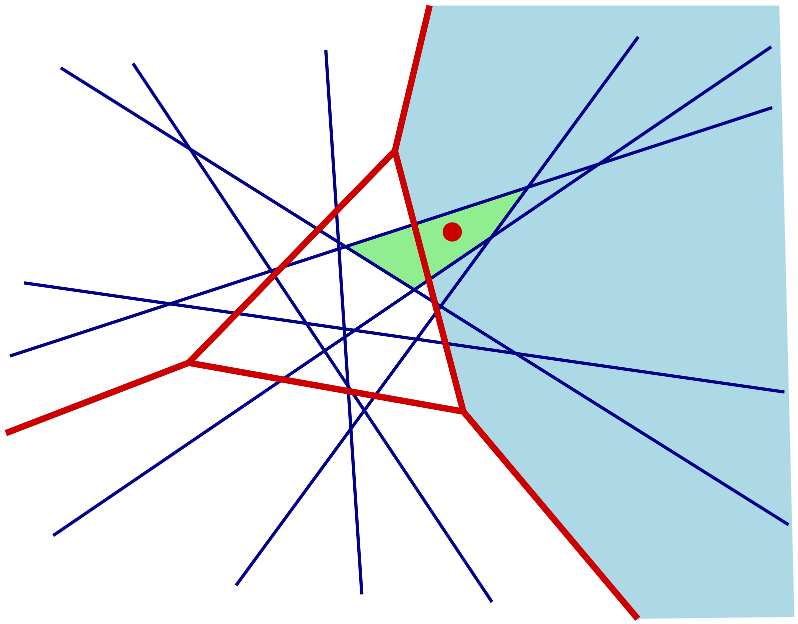
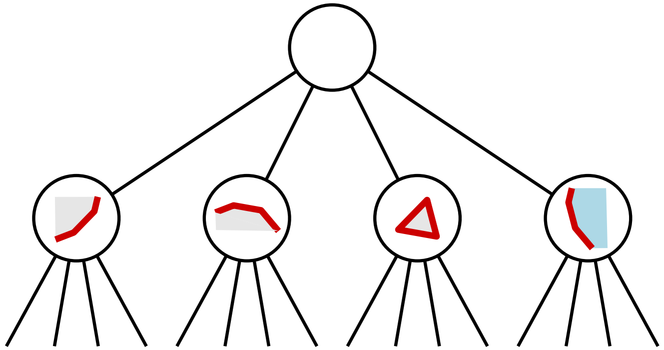




Store lines in data structure such that we can find the cell containing a query point q in $O(\log n)$ time





Analysis

Suppose each region intersects at most n/r lines, for some constant r

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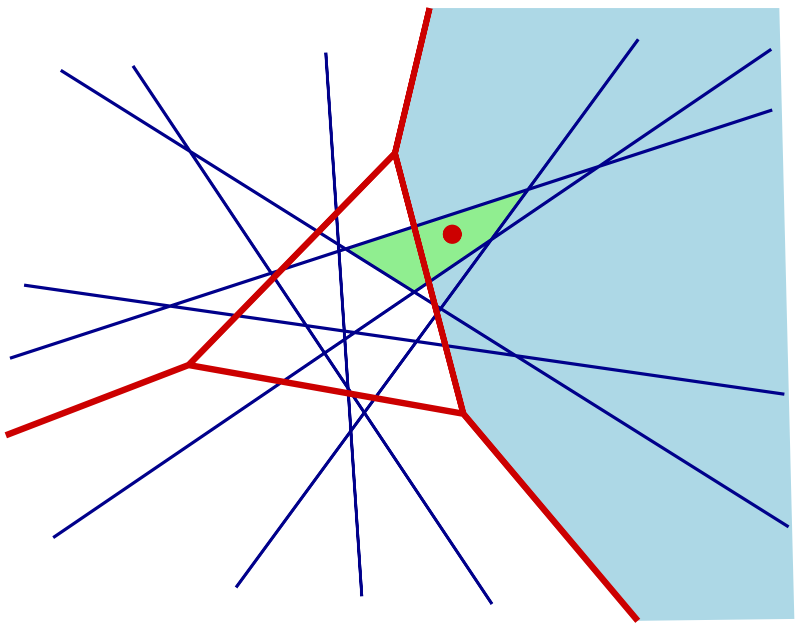

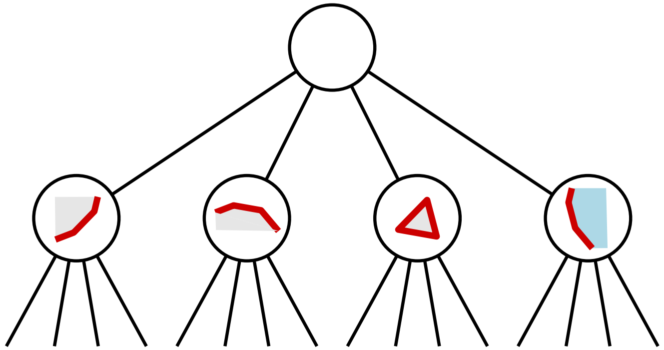

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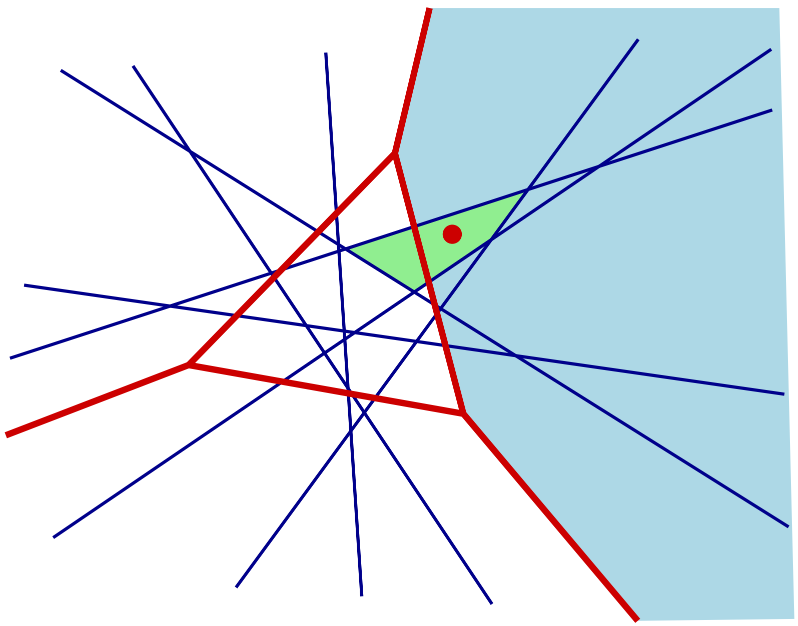

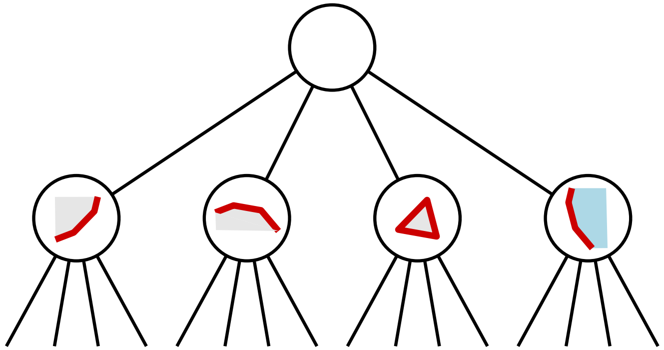

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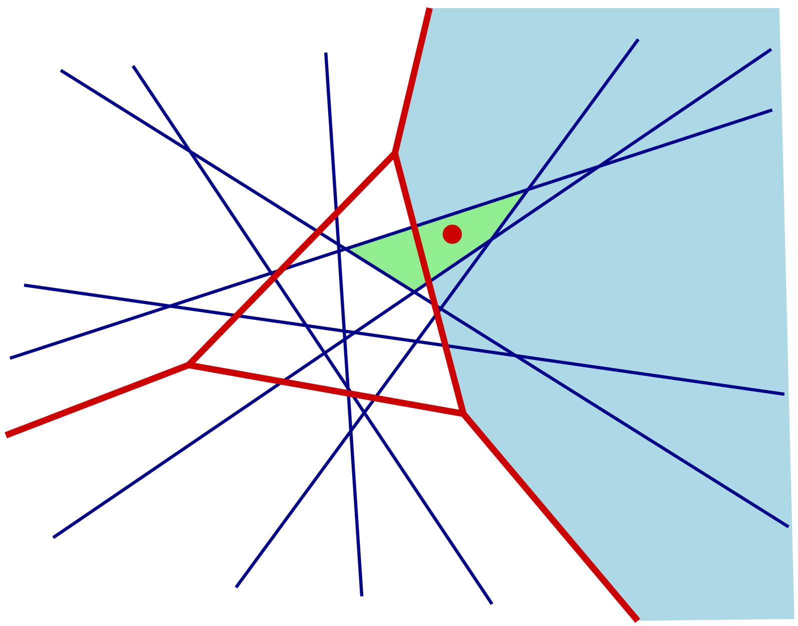

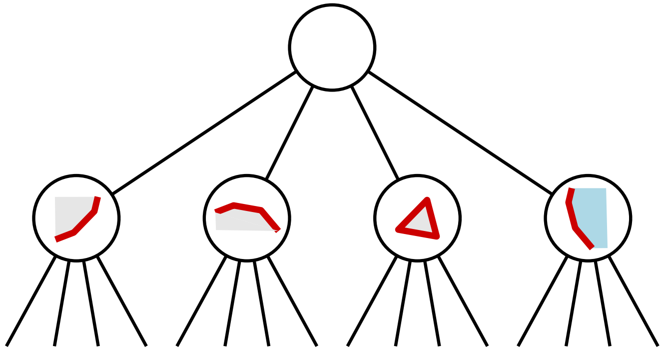

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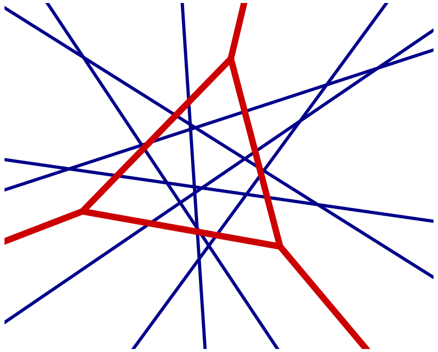
$(1/r)$-cutting for set L of n lines in $\mathbb{R}^2$

partitioning of $\mathbb{R}^2$ into (possibly unbounded) triangles Δ_i such that each Δ_i intersects only n/r lines



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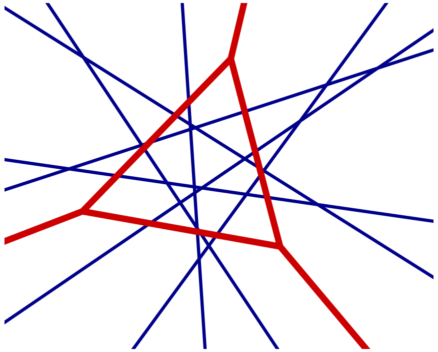

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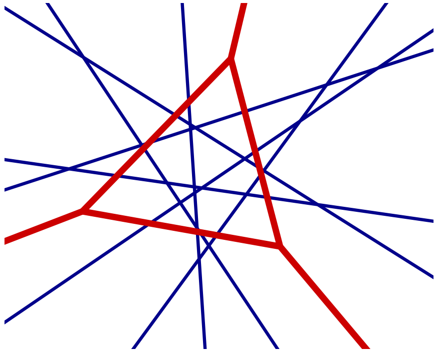

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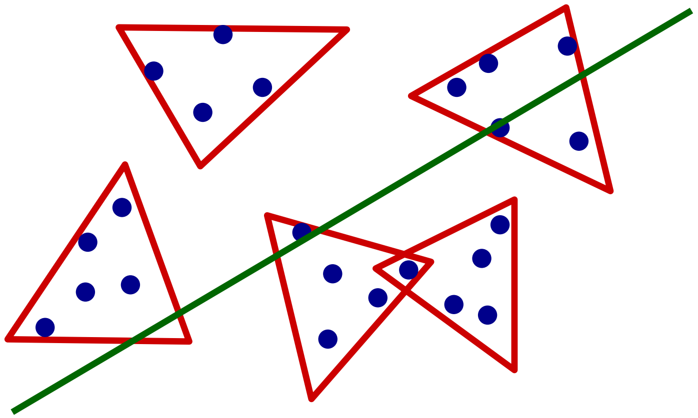
Theorem. For any set L of n hyperplanes in $\mathbb{R}^d$ and any r with $1 \leqslant r \leqslant n$ there is a $(1/r)$-cutting consisting of $O(r^d)$ simplices.

fine simplicial partition for set P of n points in $\mathbb{R}^2$

collection $\{(P_1, \Delta_1), \ldots, (P_r, \Delta_r)\}$ where

- $P = P_1 \cup \ldots \cup P_r$ and P_i's are disjoint
- Δ_i is triangle containing P_i
- $n/(2r) \leqslant |P_i| \leqslant 2n/r$



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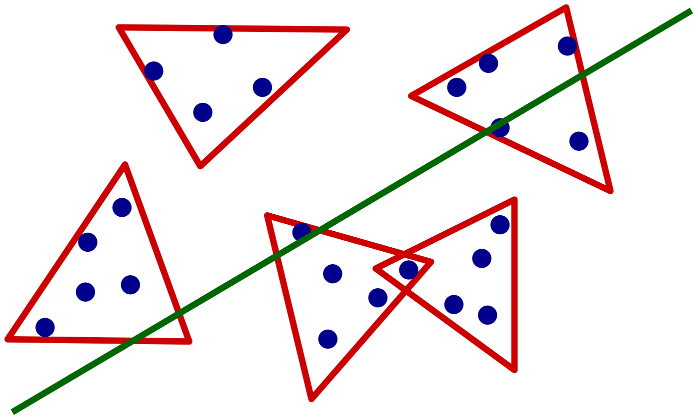

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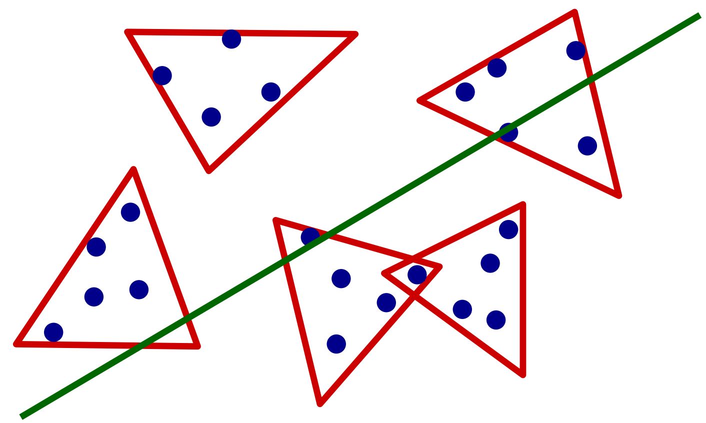

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Partitioning Arrangements: Basic Results

Cuttings (and simplicial partitions) form the basis of data structures for

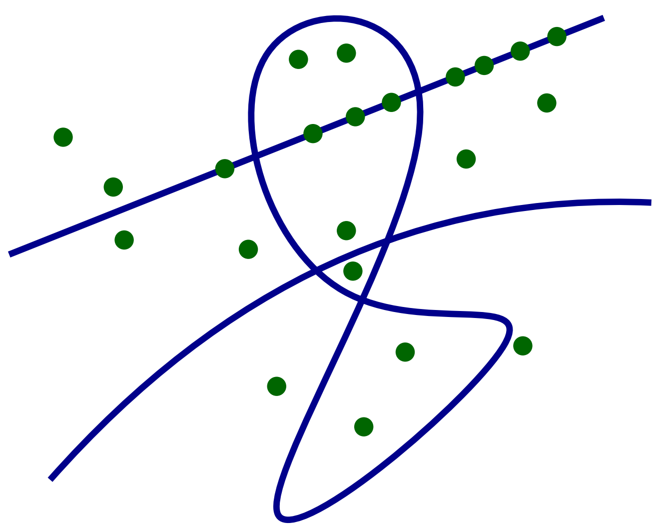
- point location
- range searching
- ray shooting

and of many other algorithmic and combinatorial results

Basic polynomial partitions [Guth-Katz'10]

- P = set of n points in $\mathbb{R}^d$
- D = parameter (can depend on n)

Theorem. There exists a surface $Z(f)$ that is the zero-set of a polynomial of degree at most D such that $\mathbb{R}^d \setminus Z(f)$ consists of $O(D^d)$ cells each containing $O(n/D^d)$ points from P.



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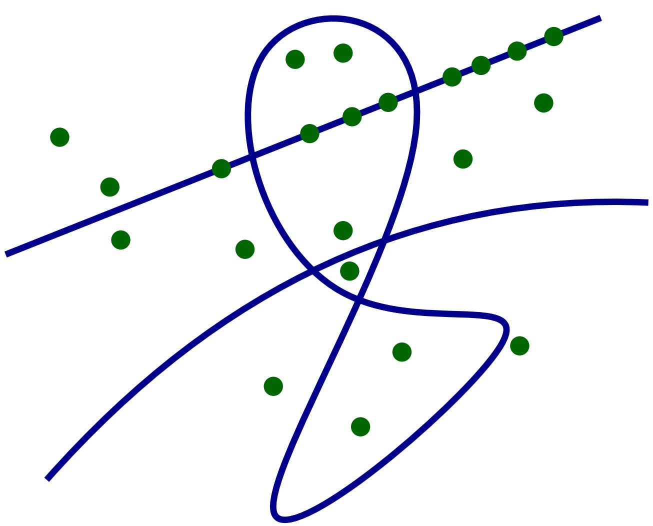
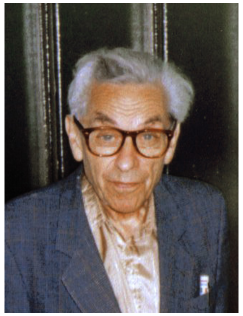


Used to (basically) solve

Erdős distinct-distances problem:

any set of n points in the plane defines $\Omega(n/\log n)$ distinct distances



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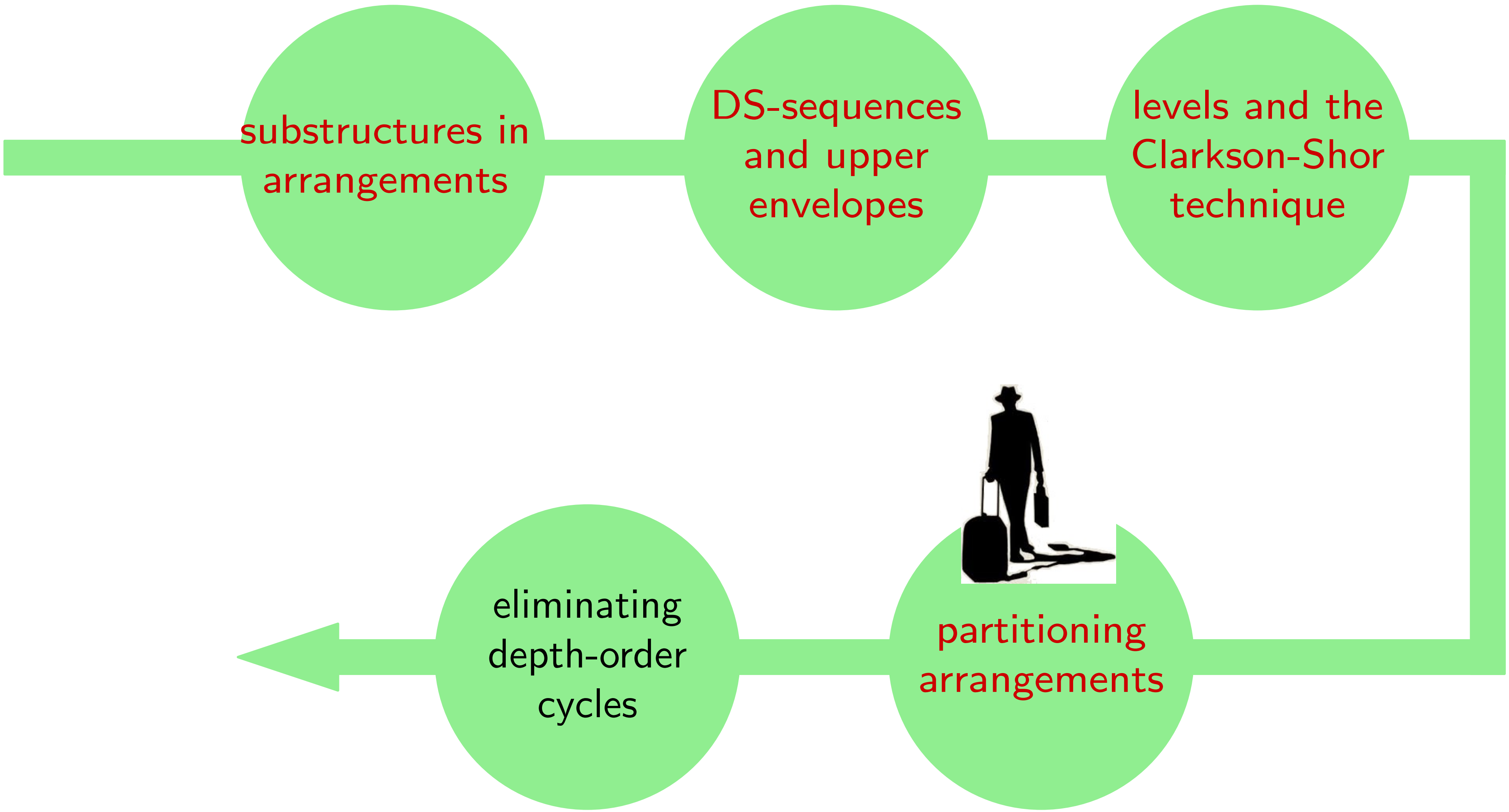
(result is actually even more general)

Lecture Overview



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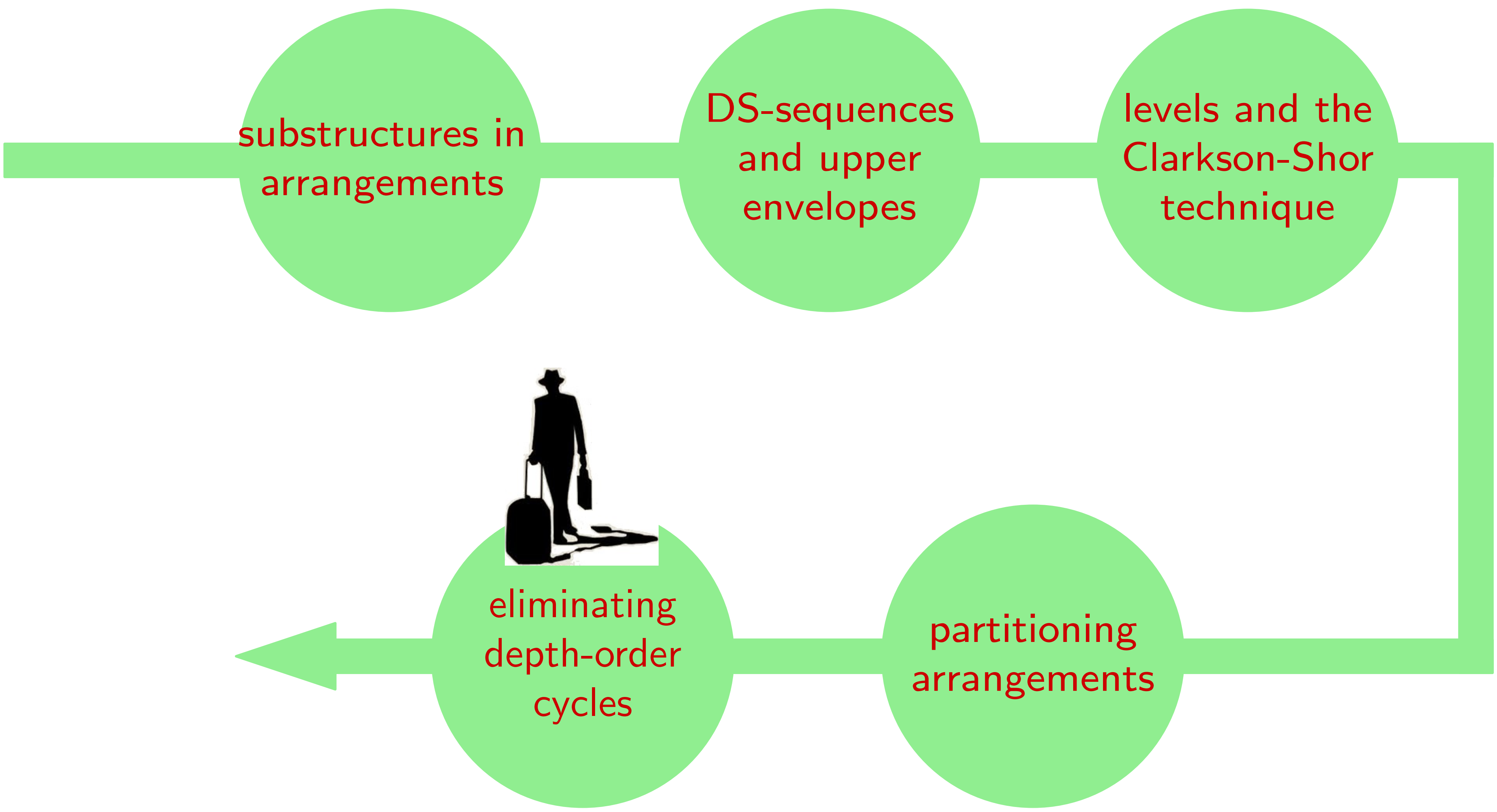
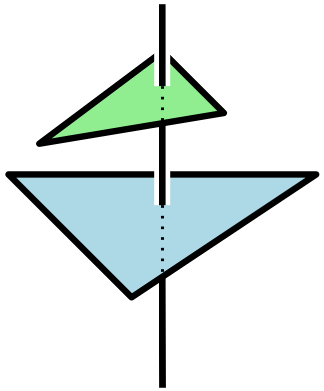


Depth orders

S: set of n disjoint triangles (or other objects) in $\mathbb{R}^3$

T is below T' (notation: $T \prec T'$):

there is a vertical line ℓ such that
$\ell \cap T$ has smaller z-coordinate than $\ell \cap T'$



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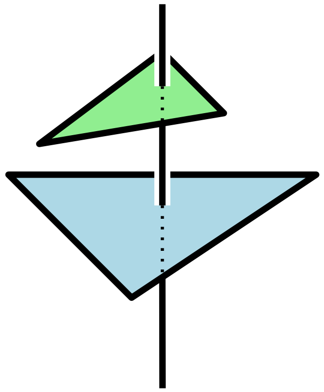

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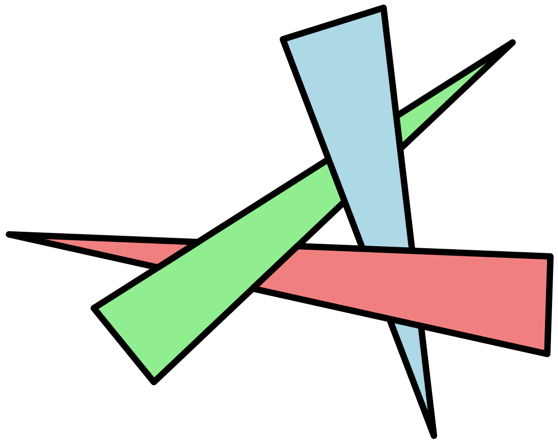
Applications

- computer graphics (Painter's Algorithm)
- computer-aided design and manufacturing (assembly sequences)

Depth order need not exist, due to cyclic overlap



Questions:

- Decide if a given order $T_1, \ldots, T_n$ is a valid depth order.
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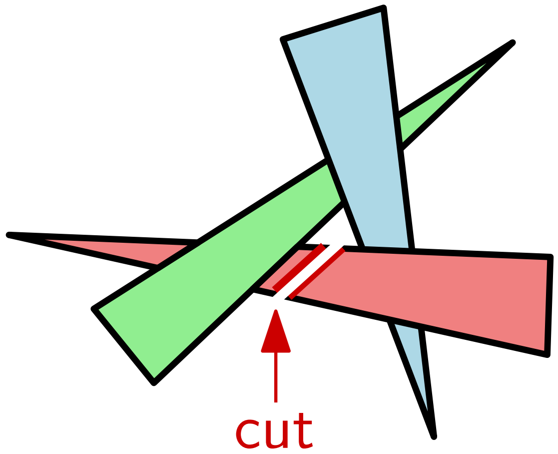

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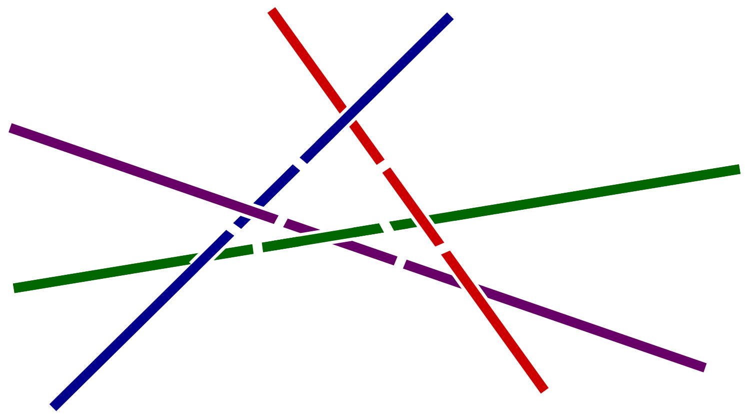

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The number of cuts to eliminate all cycles

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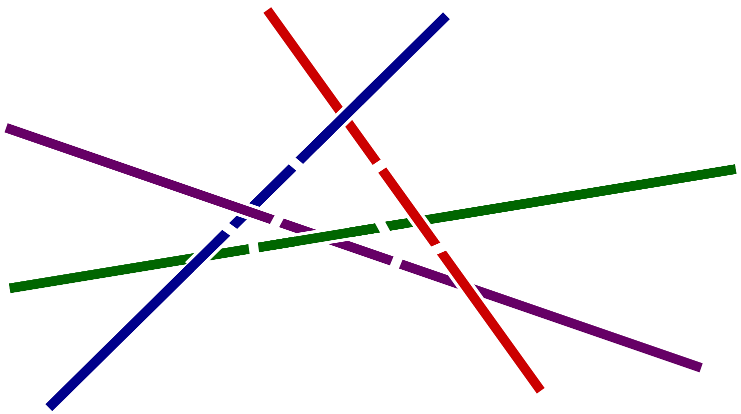

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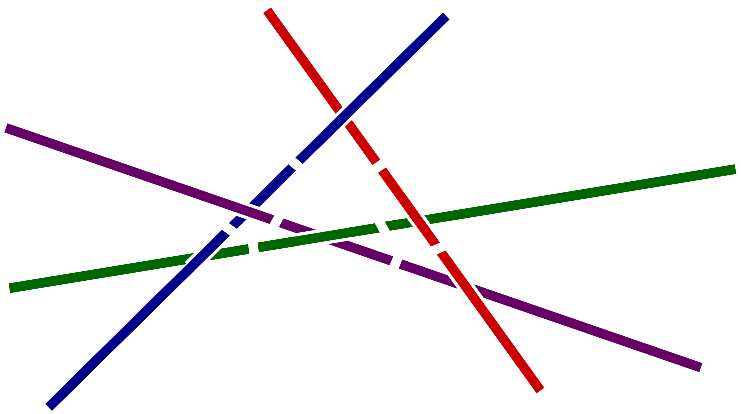

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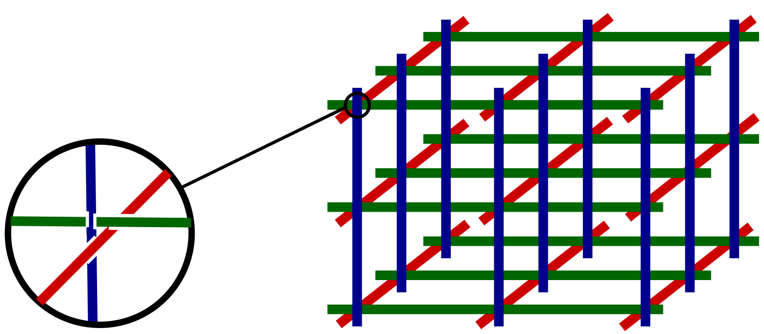

three groups of $\sqrt{n} \times \sqrt{n}$ segments each

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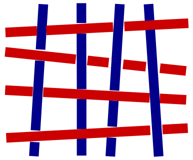

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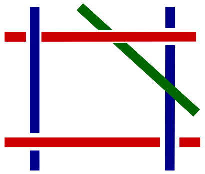

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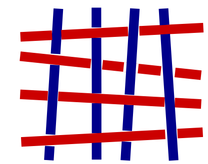

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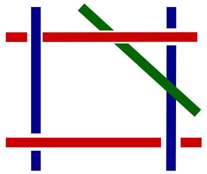

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- $O(n^{3/2}\operatorname{polylog} n)$ for line segments
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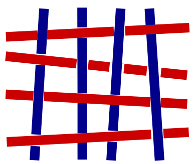

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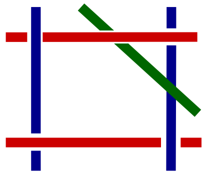

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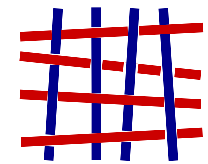

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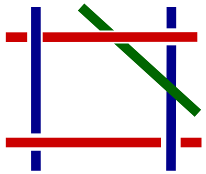

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uses curved cuts

- $O(n^{7/4} \operatorname{polylog} n)$ for triangles,
  with straight-line cuts [dB, FOCS'18]

combines Aronov-Sharir result
with cuttings

Polynomial Partitions for Lines in $\mathbb{R}^3$

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- $L =$ set of n lines, or line segments, in $\mathbb{R}^3$
- $D =$ parameter

Theorem. There exists a surface $Z(f)$ that is the zero-set of a polynomial of degree at most D such that $\mathbb{R}^d \setminus Z(f)$ consists of $O(D^3)$ cells each intersecting $O(n/D^2)$ lines from L.

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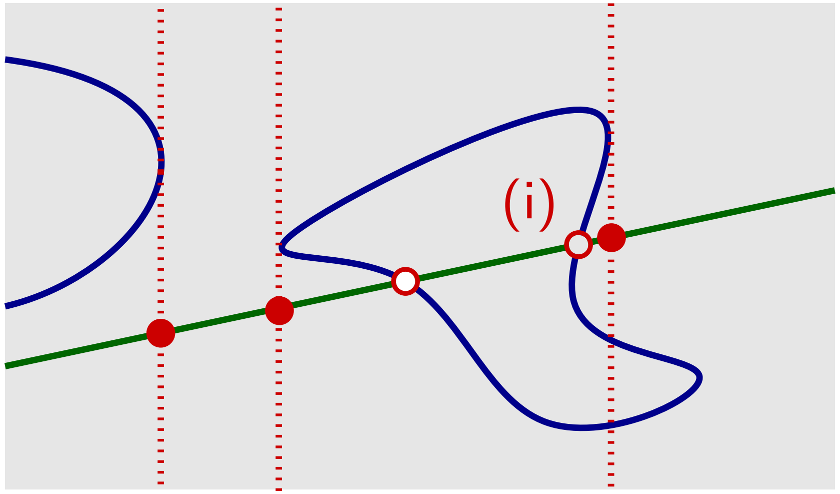
   (ii) Take vertical plane $h(\ell)$ containing ℓ. Cut ℓ at points below vertical tangencies of $h(\ell) \cap Z(f)$ and points below singularities.

   $O(D^2)$ cuts



○ type (i) cut

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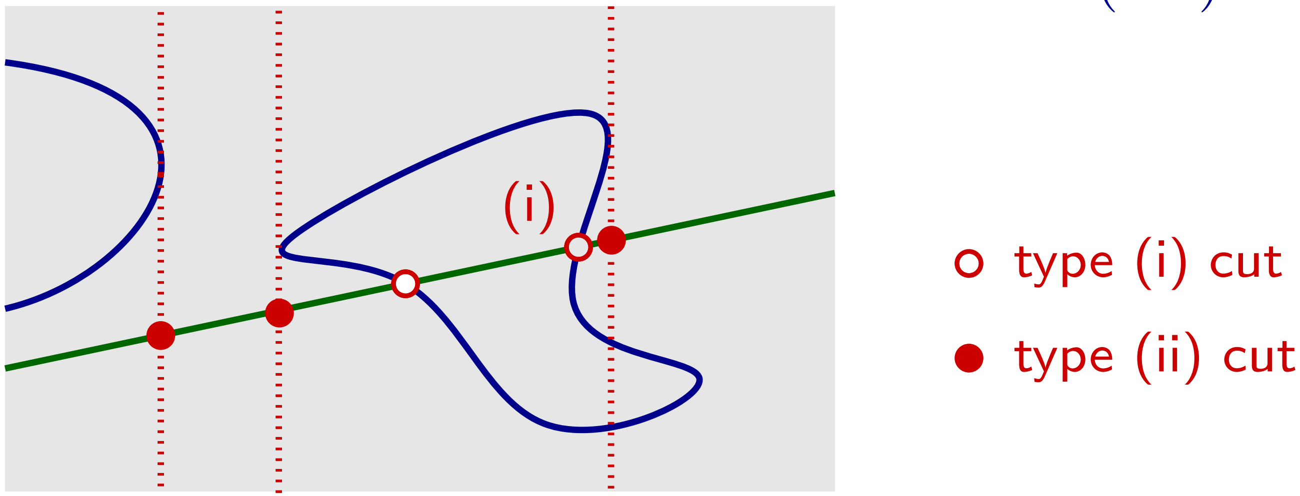

3. Recursively cut lines within each cell.

Lemma. Procedure eliminates all cycles.

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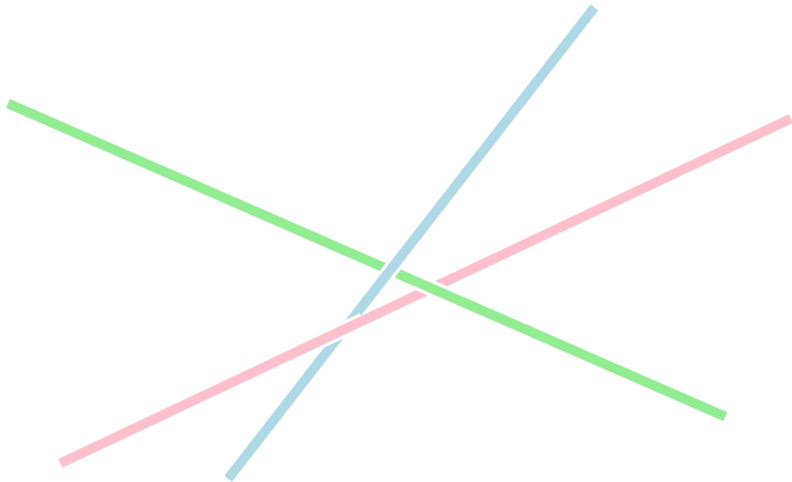
Proof. Suppose for a contradiction that there is still a cycle



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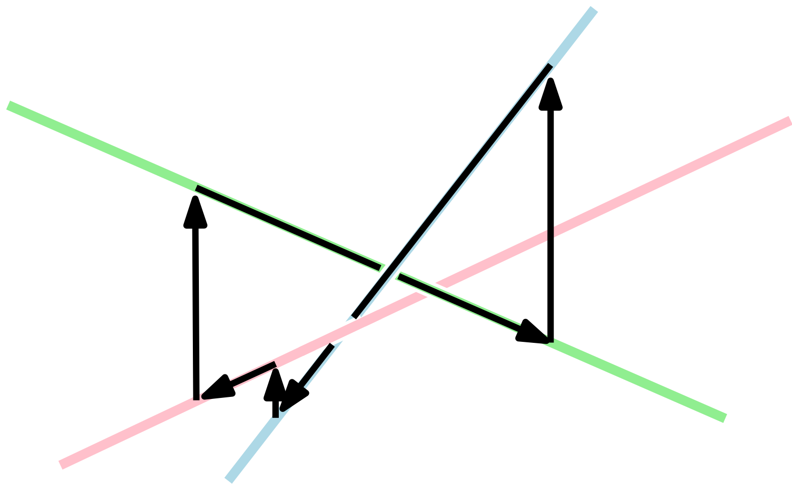

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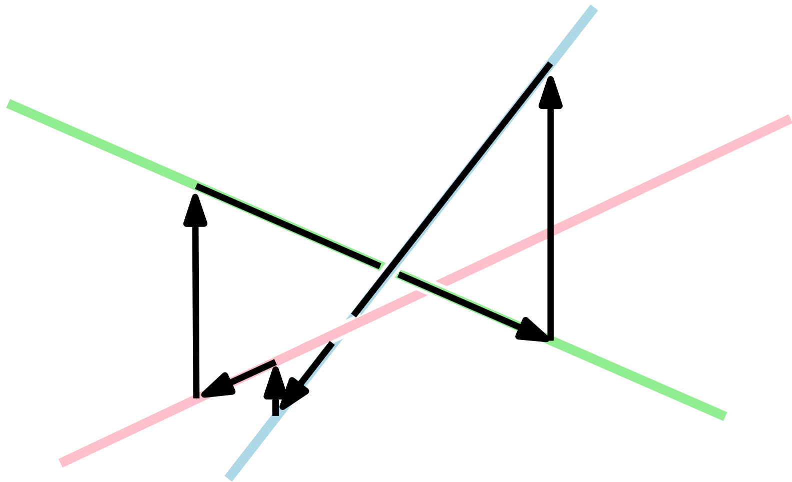
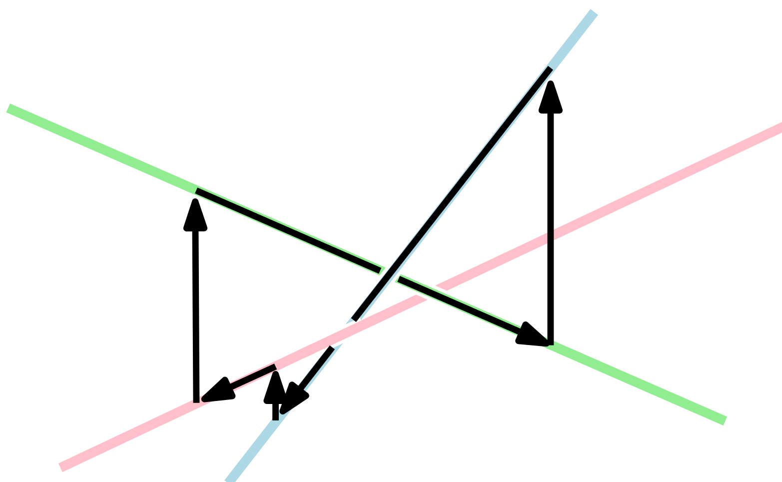


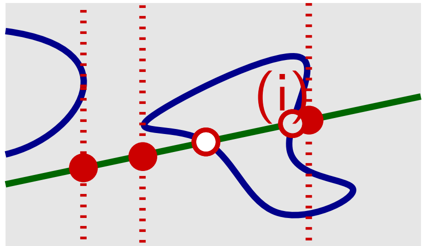
- associate 3D polygonal curve Γ to cycle
- if Γ lies completely inside cell of $\mathbb{R}^3 \setminus Z(f)$ then cycles are removed by induction

Lemma. Procedure eliminates all cycles.

Proof. Suppose for a contradiction that there is still a cycle



- associate 3D polygonal curve Γ to cycle
- if Γ lies completely inside cell of $\mathbb{R}^3 \setminus Z(f)$ then cycles are removed by induction
- otherwise consider how level (number of intersection of upward ray with $Z(f)$) changes as we walk along Γ



The Aronov-Sharir approach for line segments

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- similar but much more complicated approach works for triangles
- triangles are cut by polynomial, so pieces have curved boundaries

Avoiding curved cuts for triangles

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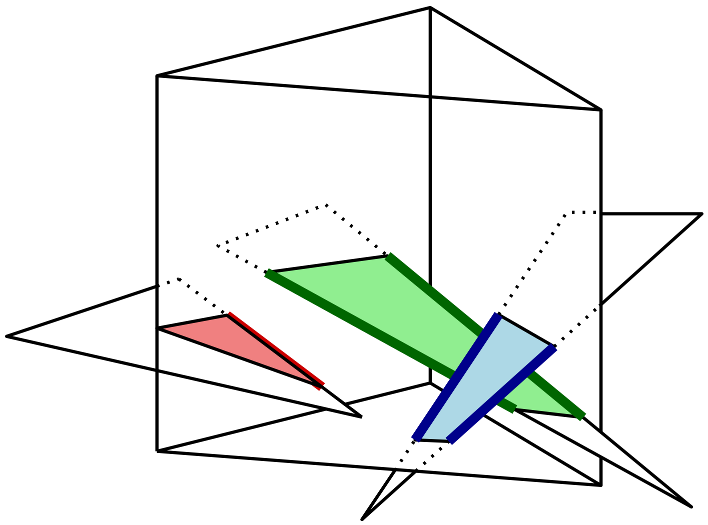
- $\mathcal{T}$ = set of n triangles in $\mathbb{R}^3$
- $\mathcal{E}$ = set of $3n$ edges of the triangles
- C = vertical, triangular column



Lemma. If C does not contain any triangle vertex in its interior and $\mathcal{E} \cap C$ is acyclic, then $\mathcal{T} \cap C$ is acyclic.

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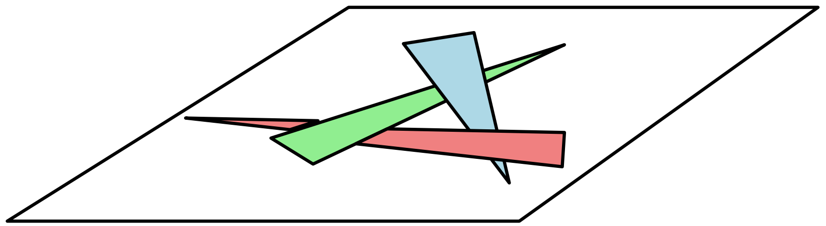

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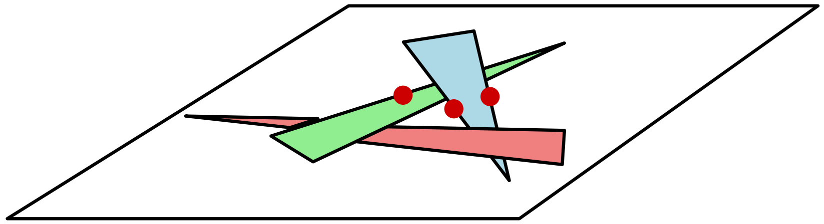

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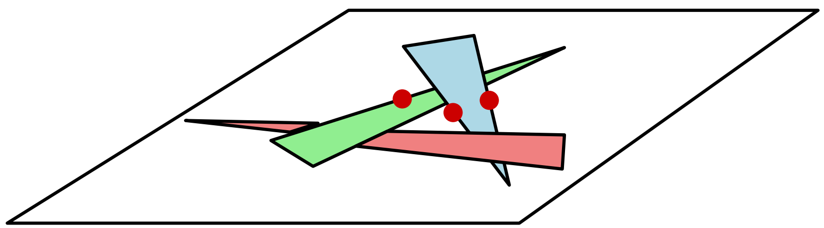

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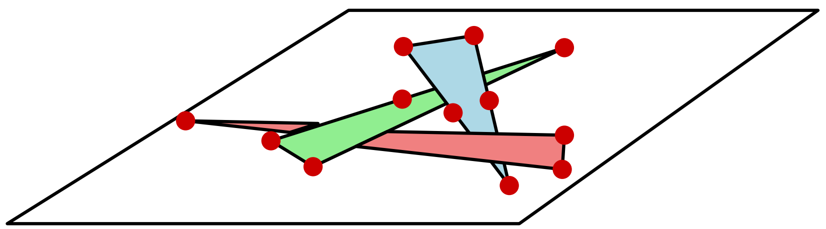

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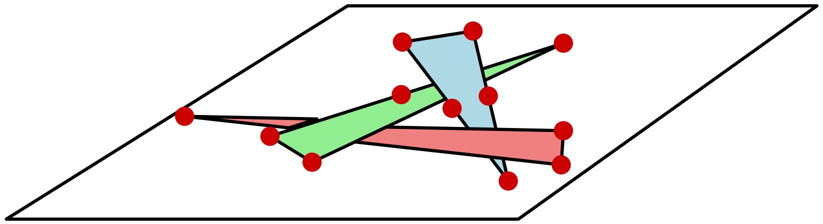

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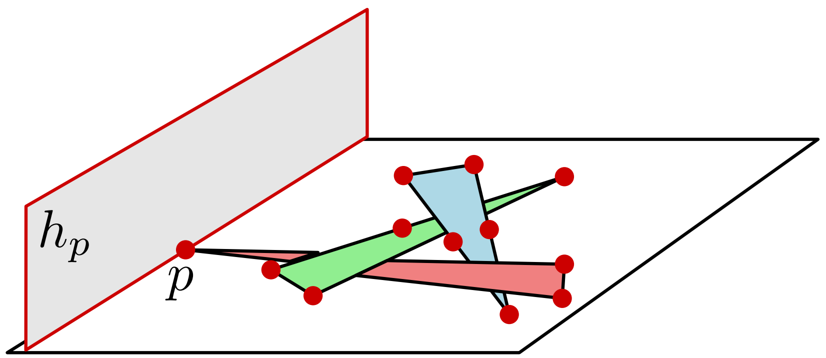

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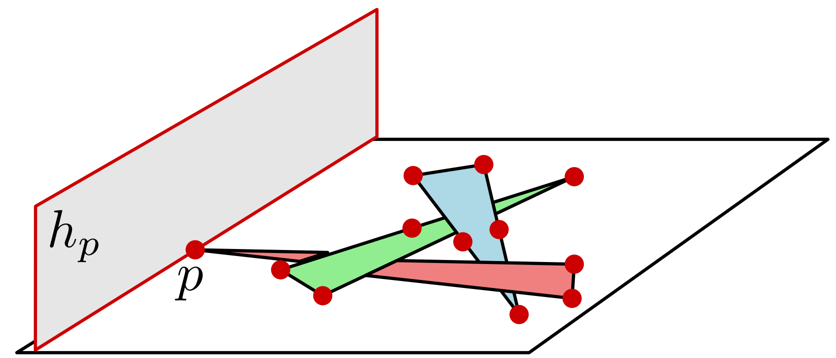

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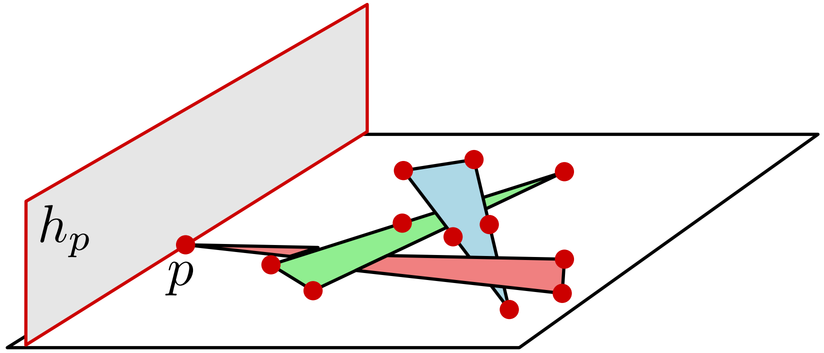

but may result in $O(n^{2.5} \operatorname{polylog} n)$ fragments.

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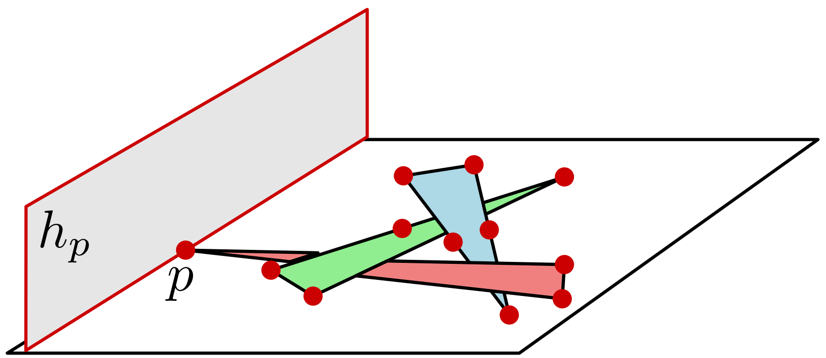

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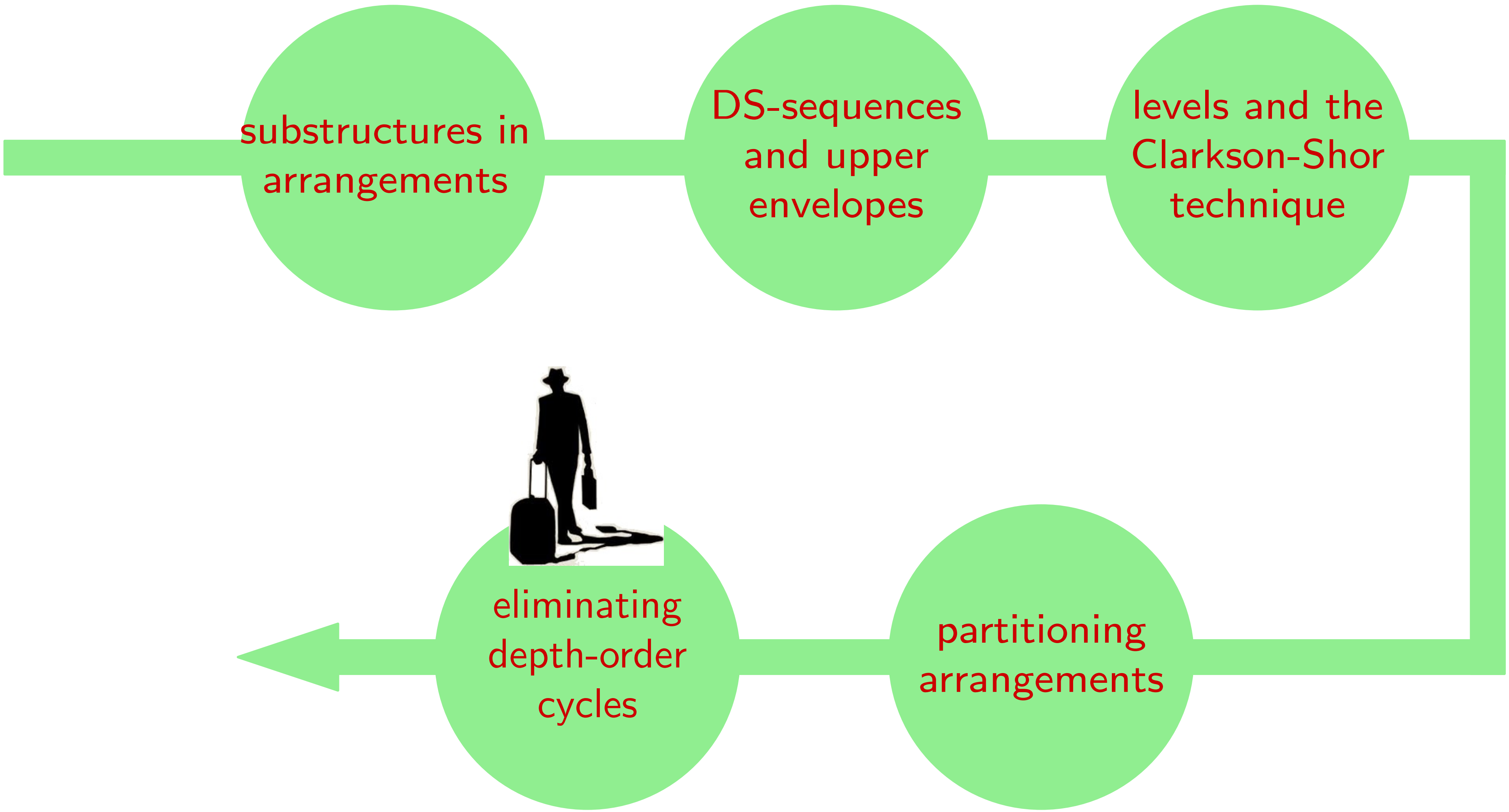
using hierarchical gives $O(n^{7/4} \operatorname{polylog} n)$ fragments.

Lecture Overview



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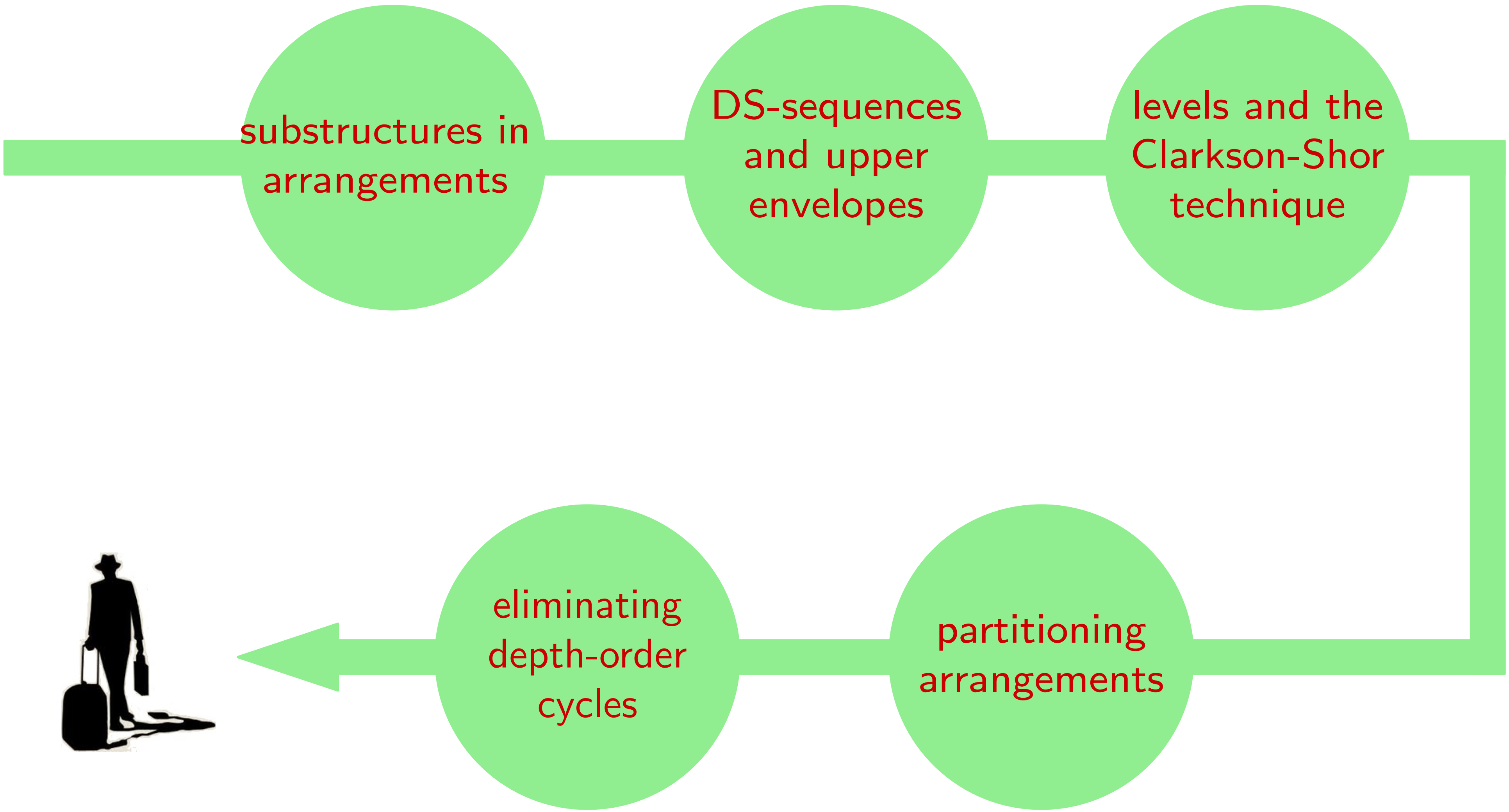
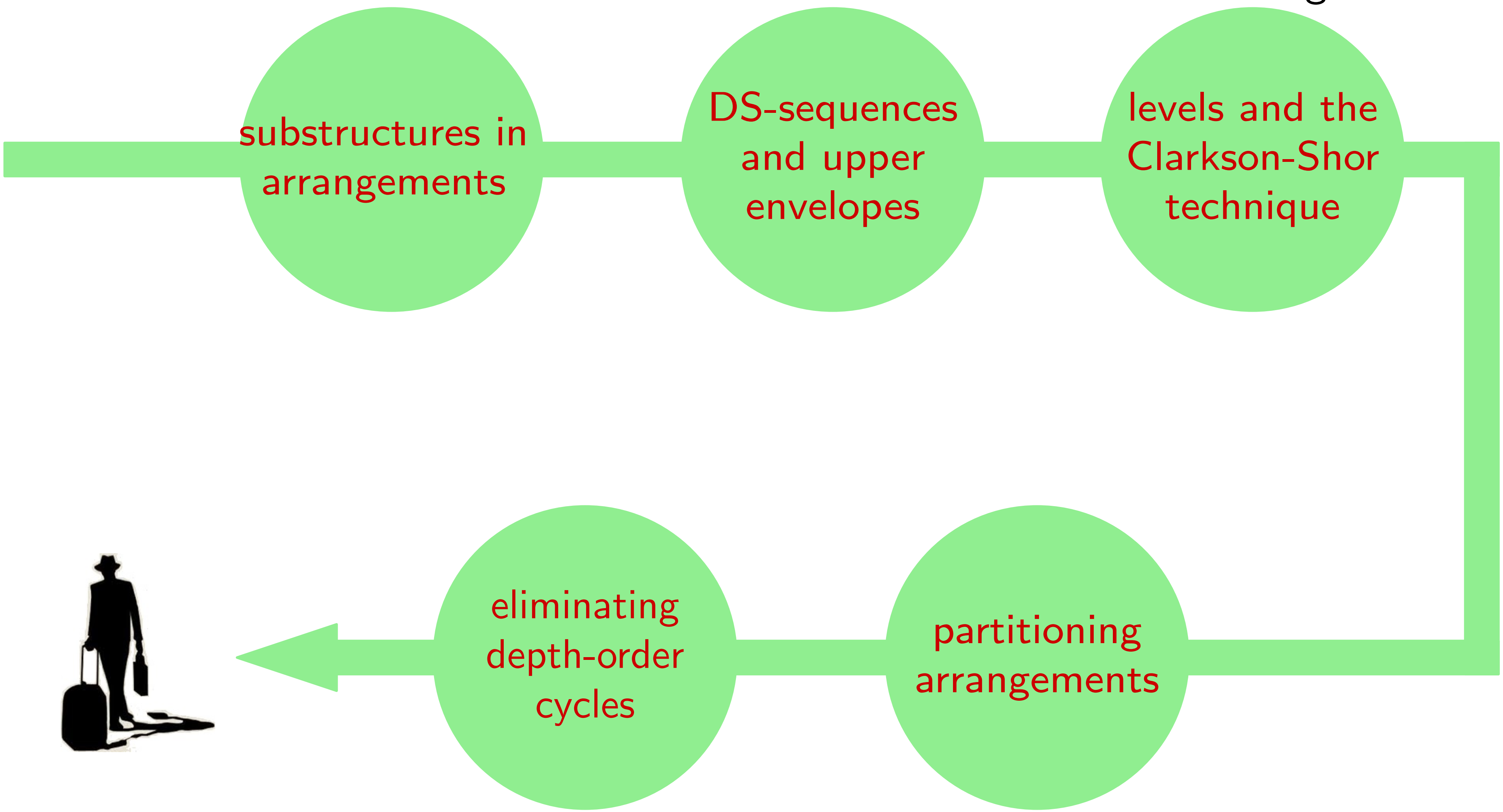
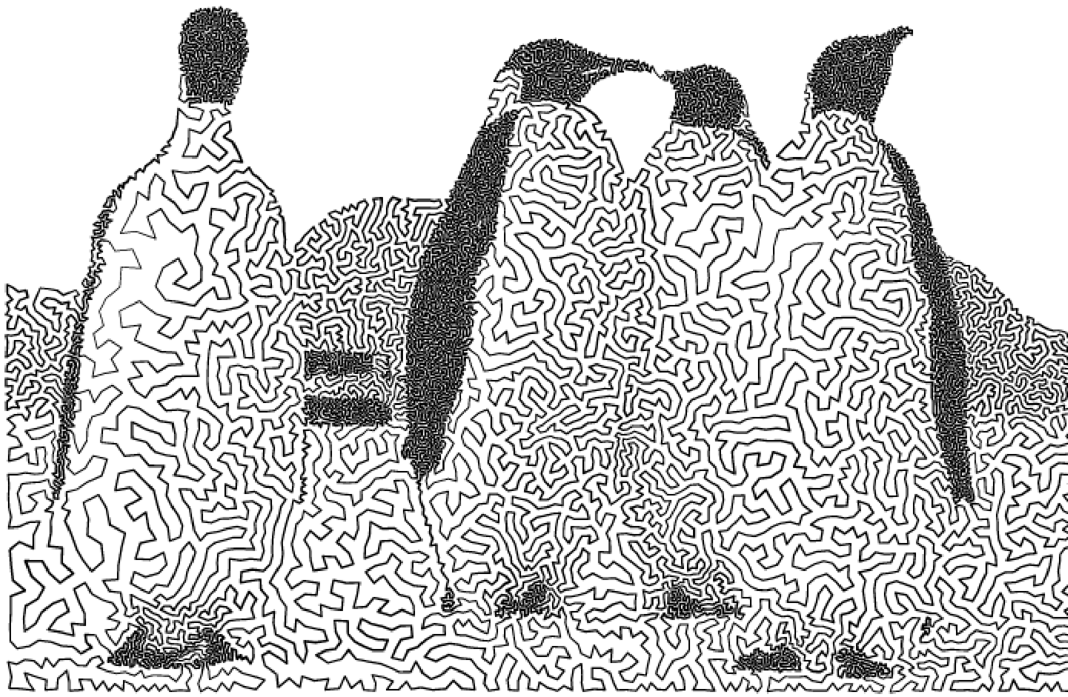


open problem:
complexity of k-level
in line arrangement



open problem:
eliminate cycles in triangles with
$o(n^{7/4})$ straight-line cuts

Thanks for your attention!



TSP Art by Carig Kaplan and Robert Bosch