Graph Coloring Problems

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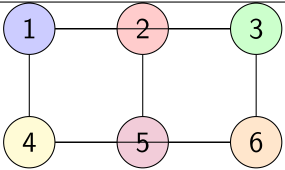
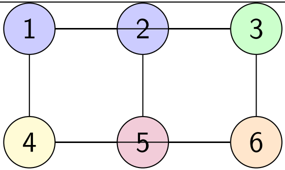
Proper vs Improper Graph Coloring

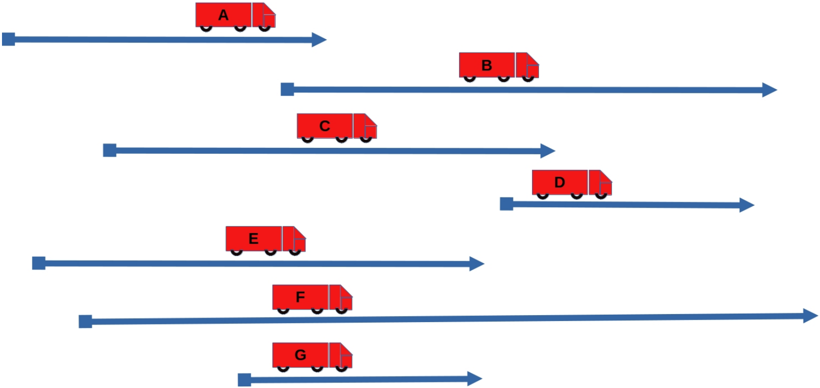
	Proper Graph Coloring	**Improper Graph Coloring**
Definition	No two adjacent vertices have the same color	Two or more adjacent vertices have the same color
Example		
Applications	Many practical applications such as scheduling, frequency assignment and register allocation	Not applicable for many practical problems as it does not satisfy the condition of no adjacent vertices having the same color.

Map Coloring

Figure 1: Map Coloring (Source :- Google Search)[1]

[1] https://commons.wikimedia.org/wiki/File:Indiamapur.jpg#/media/File:Indiamapen.svg/

Resource Allocation



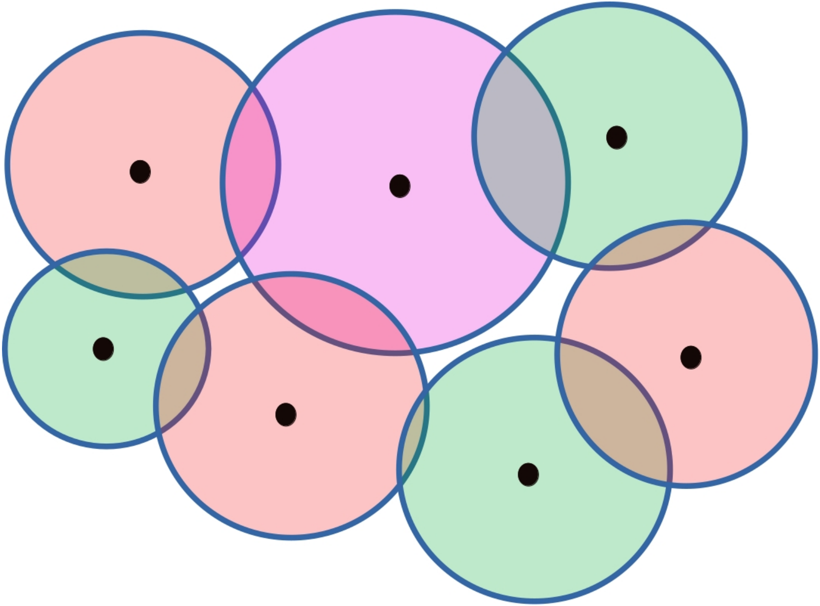
Timetabling and Scheduling

Time schedule for classes

Day	08:30-09:50	10:00-11:20	11:30-12:50	13:00-14:20	14:30-15:50	16:00-17:20
Monday	IC251 (14) CA200 (56)	CS102 (14) CS201 (56) EE101(14) ME212 (16)	CS204 (36) DS100 (36) EE207 (36) ME231 (16)		CS202 (14) EE201(14) LA332 (55)	LA344 (12) LA339 (34) LA348 (56)
Tuesday	CS100 (36) MA605 (36) EE208 (36)	CS202 (14) EE201(14) LA332 (55)	CS203 (14) CS253 (56) DS200 (16) EE202 (16) ME251 (16)			
Wednesday	IC251 (14) CA200 (56)	CS102 (14) CS201 (56) EE101(14) ME212 (16)	CS204 (36) DS100 (36) EE207 (36) ME231 (16)		TUT: IC251	LA344 (12) LA339 (34) LA348 (56)
Thursday	CS100 (36) MA605 (36) EE208 (36)		CS203 (14) CS253 (56) DS200 (16) EE202 (16) ME251 (16)		CS202 (14) EE201(14) LA332 (55)	LA344 (12) LA339 (34) LA348 (56)
Friday	IC251 (14) CA200 (56)	CS102 (14) CS201 (56) EE101(14) ME212 (16)	CS204 (36) DS100 (36) EE207 (36) ME231 (16)		CS100 (36) MA605 (36) EE208 (36)	CS203 (14) CS253 (56) DS200 (16) EE202 (16) ME251 (16)

Figure 2: Timetabling (Source :- IIT Bhilai)

Frequency assignment



1	2						8	4
	3						7	
		4				6		
			2		3			
		5				9		
		6		9		5		
	7						2	
				5				

Figure 3: Sudoku Puzzle (Source :- Google Search)[2]

[2] https://commons.wikimedia.org/wiki/File:Sudoku_Puzzle_(18_clue_-_R828-S09).png

Sudoku puzzles

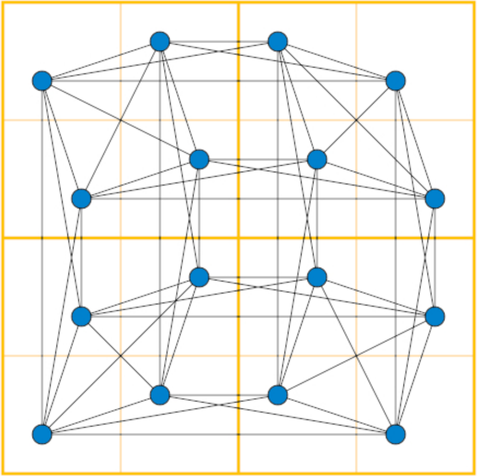


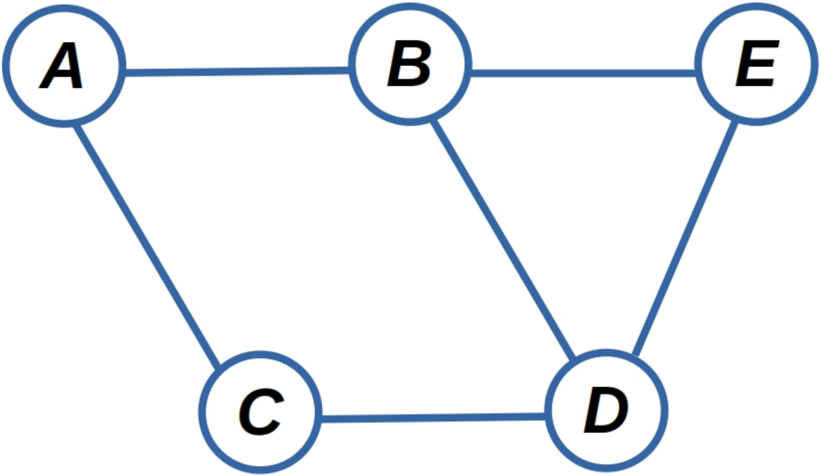
Figure 4: Sudoku Graph (Source :- Google Search)[3]

[3] https://upload.wikimedia.org/wikipedia/commons/5/5c/4x4_Sudoku_graph.svg

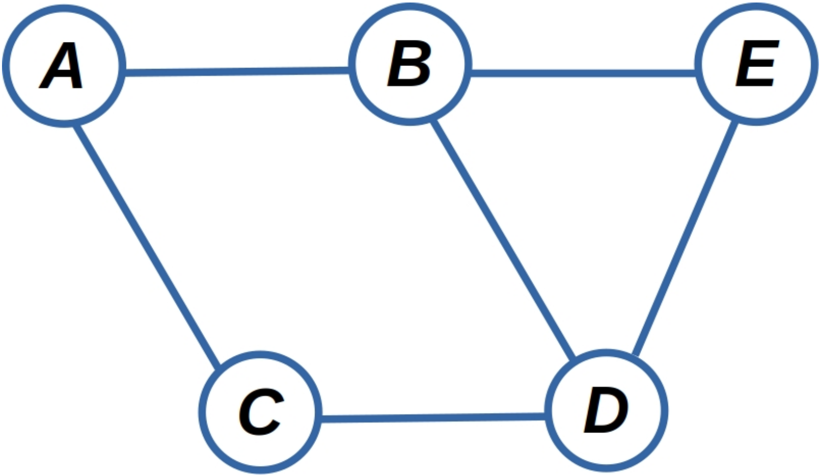
Other Applications

- Register allocation in compilers
- Round Robin Tournament Scheduling
- Flight Scheduling
- Scheduling of municipal waste collections

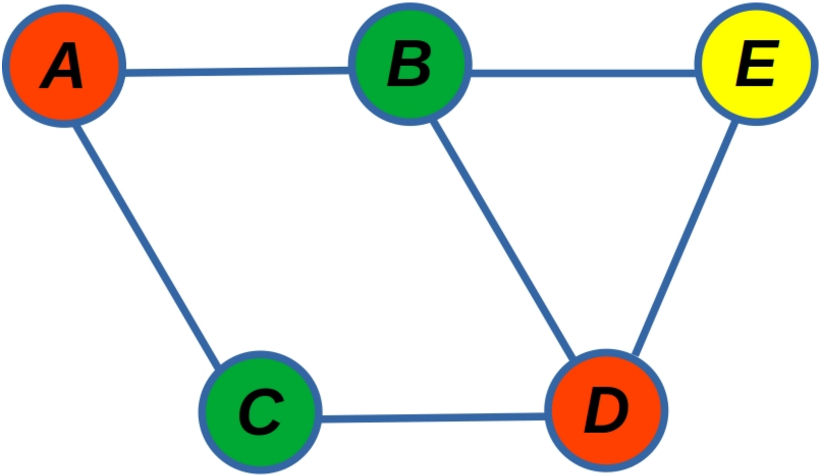
Graph Coloring



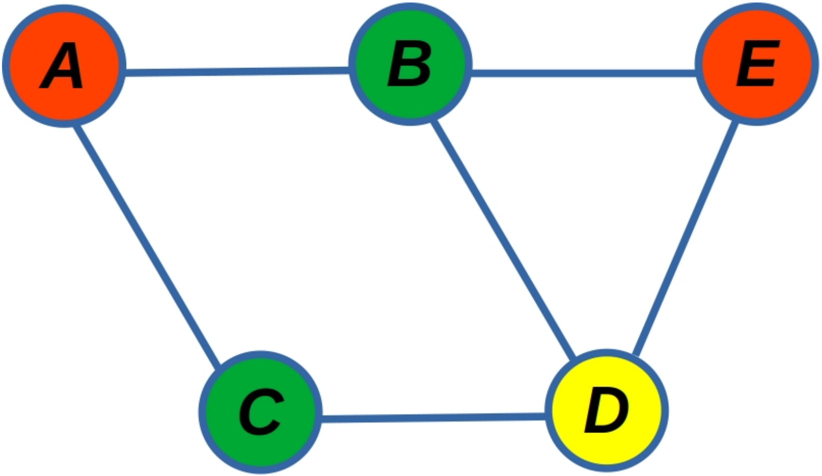
Vertex Coloring



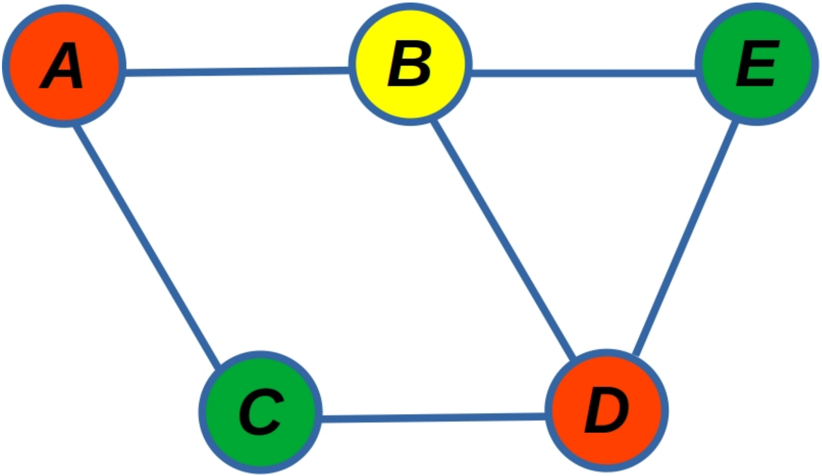
Vertex Coloring



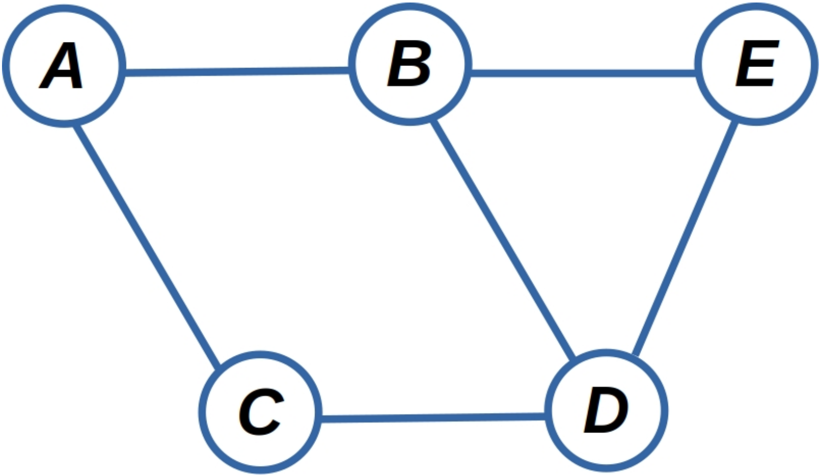
Vertex Coloring



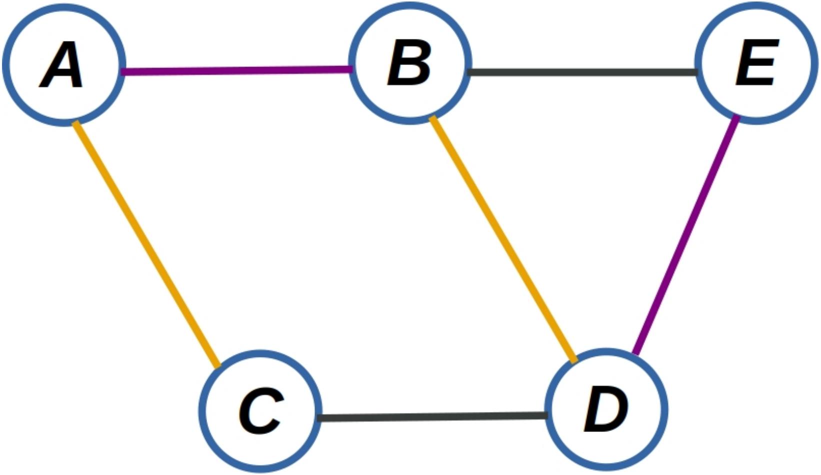
Vertex Coloring



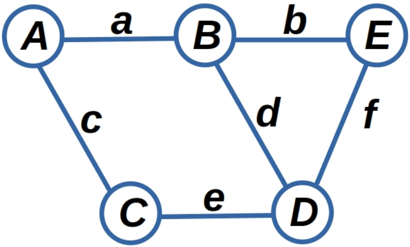
Edge Coloring



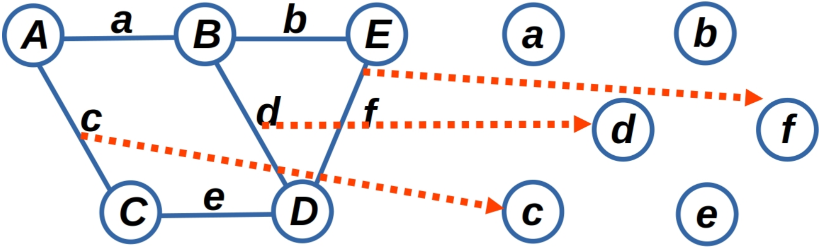
Edge Coloring



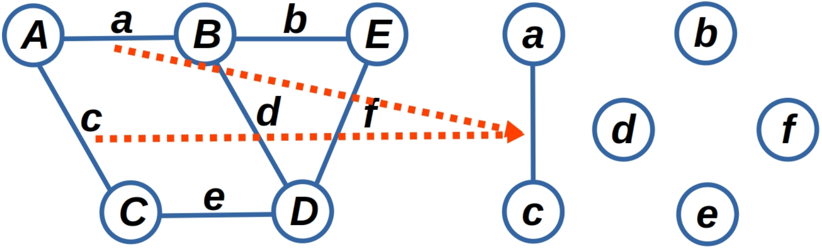
Line Graph: Relation between Edge and Vertex coloring



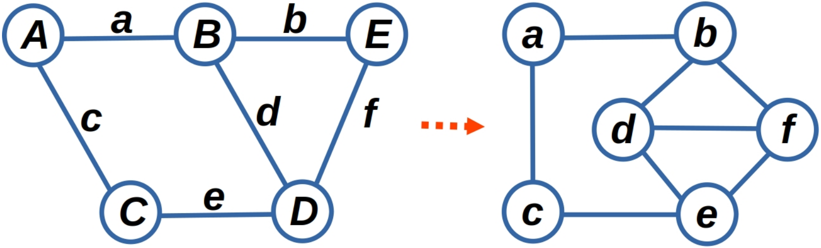
Line Graph: Relation between Edge and Vertex coloring

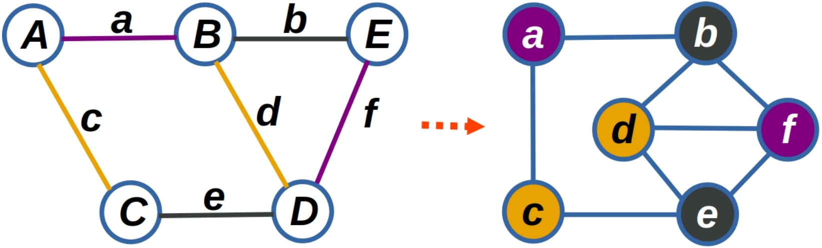


Line Graph: Relation between Edge and Vertex coloring

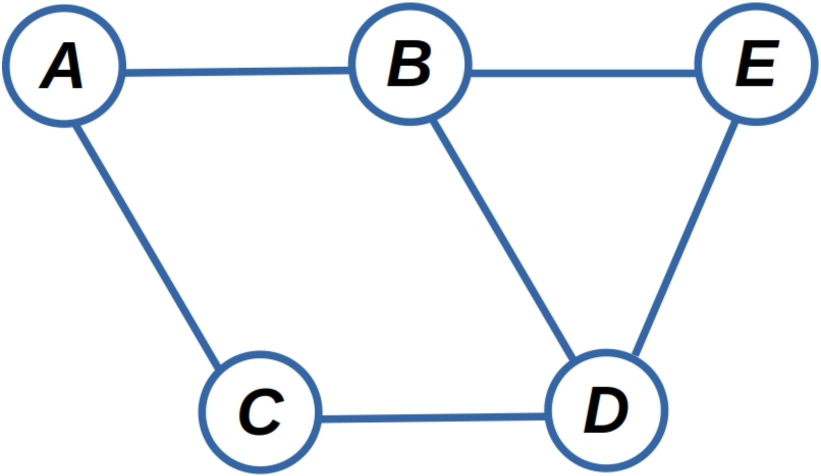


Line Graph: Relation between Edge and Vertex coloring

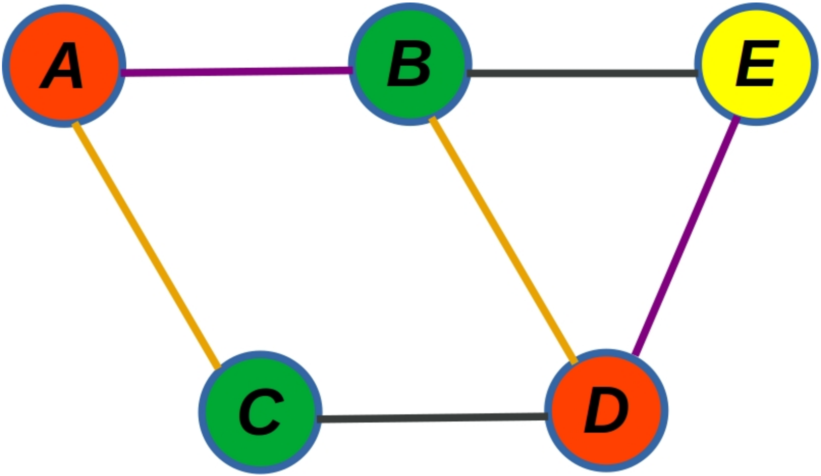



A
a
B
b
E
c
d
f
C
e
D
a
b
d
f
c
e

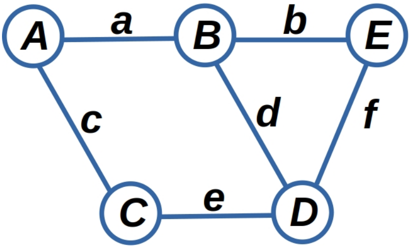
Total Coloring



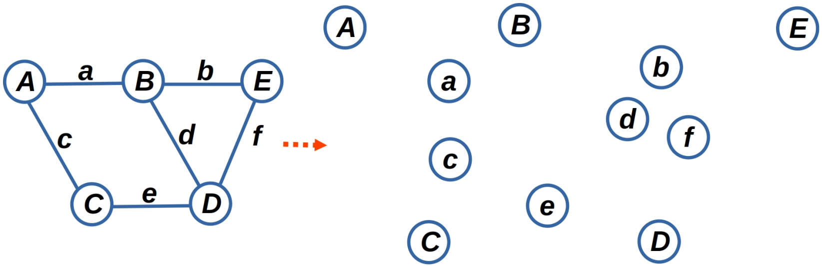
Total Coloring



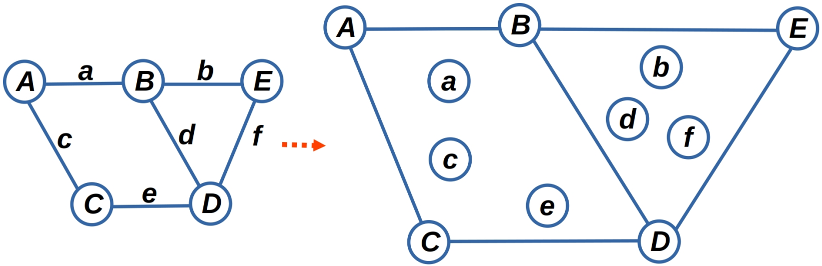
Total Graph: Relation between Total and Vertex coloring



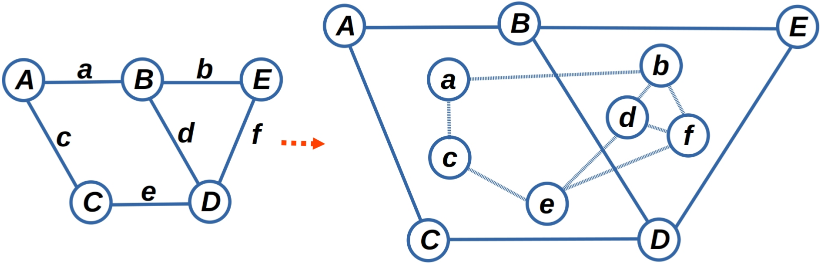
Total Graph: Relation between Total and Vertex coloring



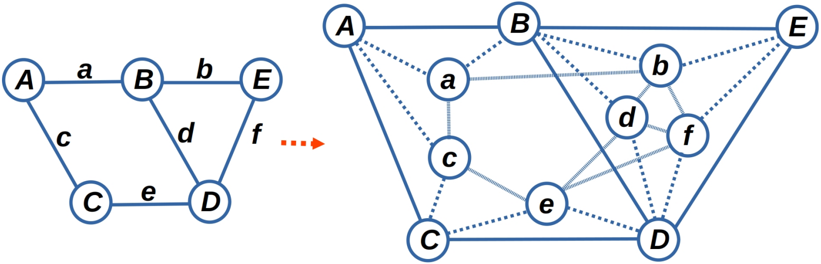
Total Graph: Relation between Total and Vertex coloring



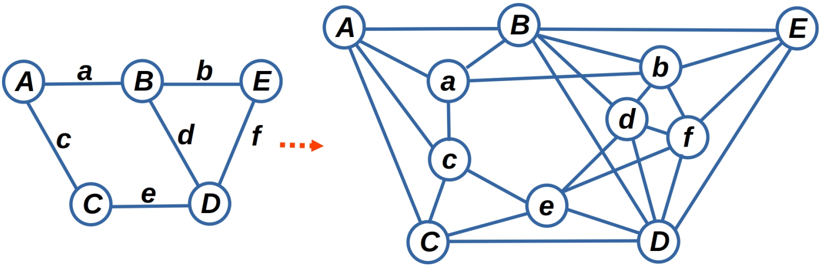
Total Graph: Relation between Total and Vertex coloring



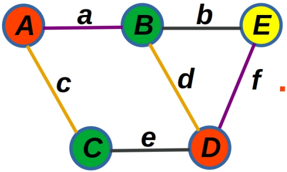
Total Graph: Relation between Total and Vertex coloring

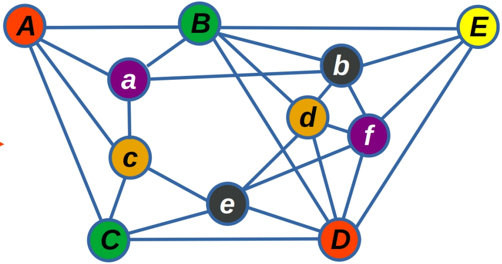


Total Graph: Relation between Total and Vertex coloring



Total Graph: Relation between Total and Vertex coloring





k-colorable, Graph Coloring Problems, and Chromatic Numbers

- k-colorable (k-COL): If a graph G can be colored with k colors.
- Graph Coloring Problems: Is a graph G k-colorable?
-
  1. $\chi(G)$: Vertex Chromatic Number
  2. $\chi'(G)$: Edge Chromatic Number (Chromatic index)
  3. $\chi''(G)$: Total Chromatic Number

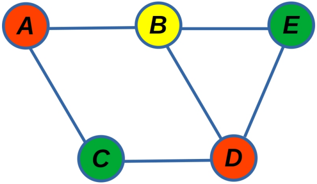
A trivial relation between $\chi''(G)$, $\chi'(G)$, and $\chi(G)$

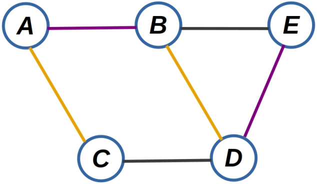
$$\chi''(G) \leq \chi'(G) + \chi(G).$$

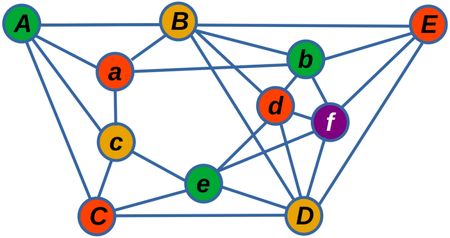
Relation

A trivial relation between $\chi''(G)$, $\chi'(G)$, and $\chi(G)$

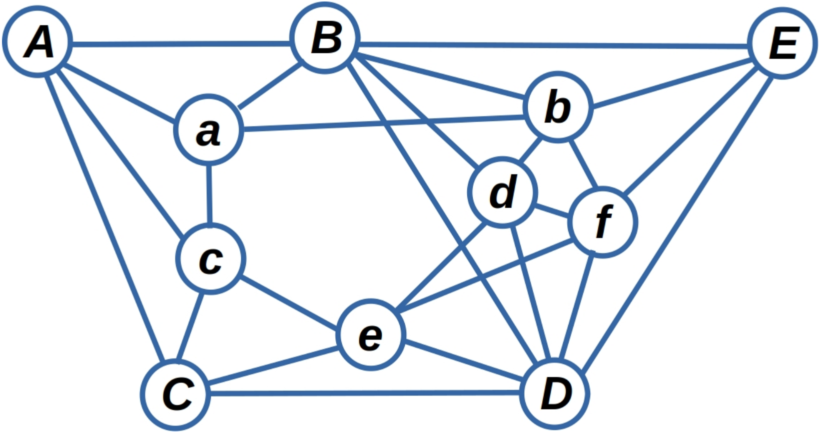
$$\chi''(G) \leq \chi'(G) + \chi(G).$$



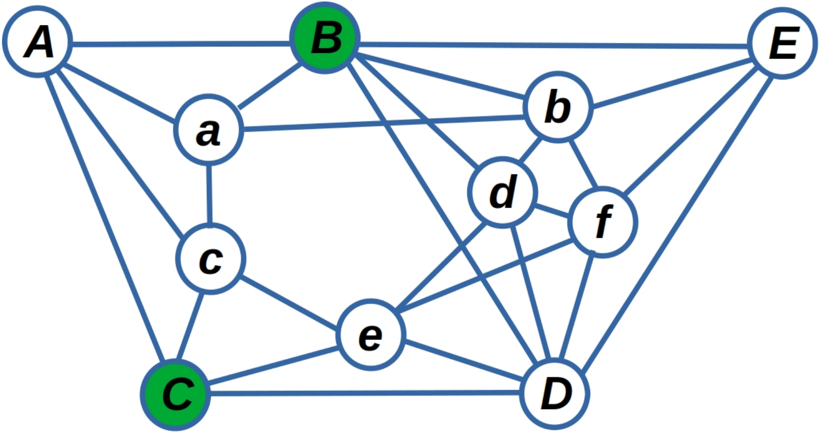




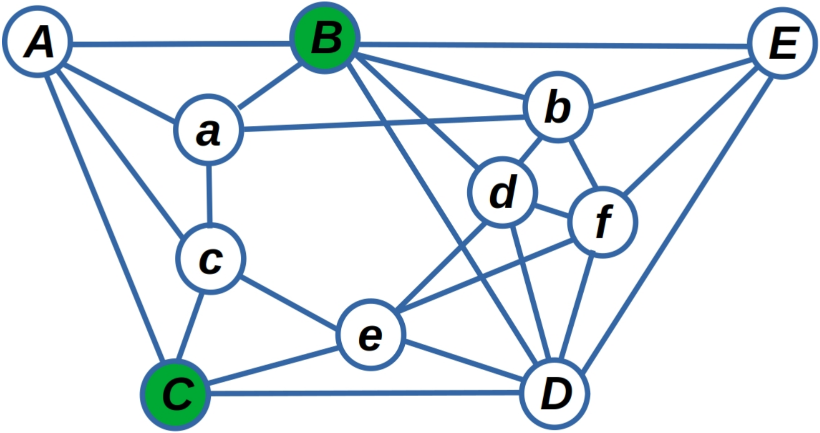
Independent Set



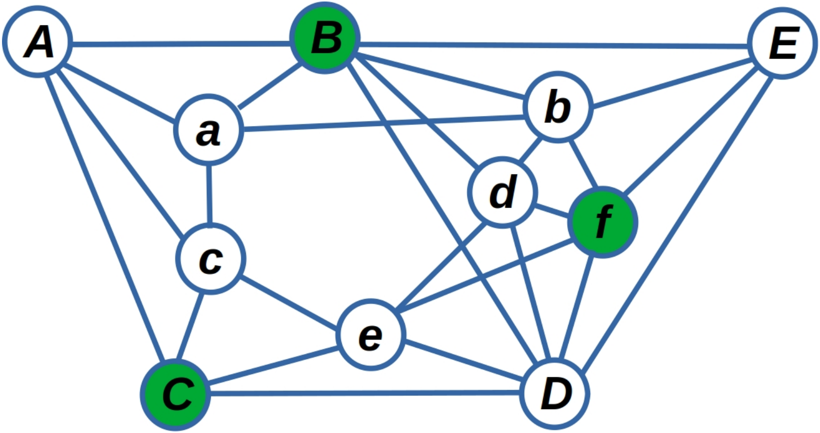
Independent Set



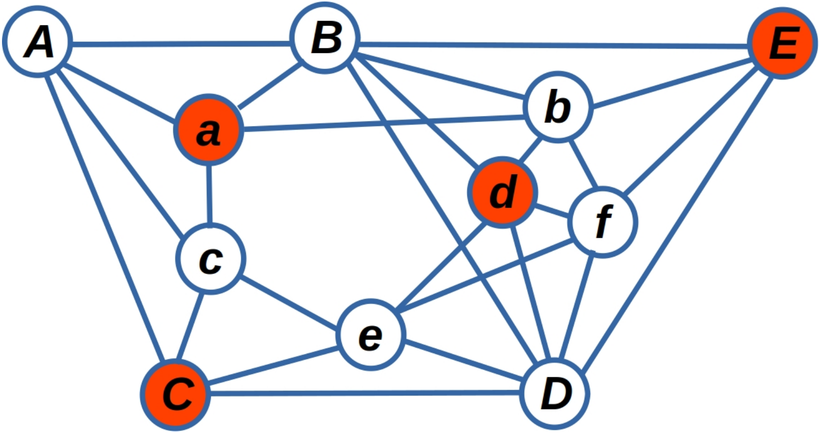
Independent Set: Is it maximal?



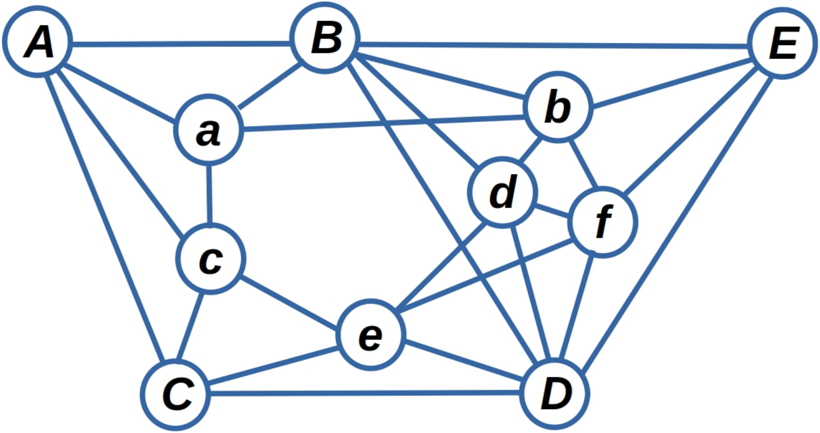
Independent Set: Is it maximum?



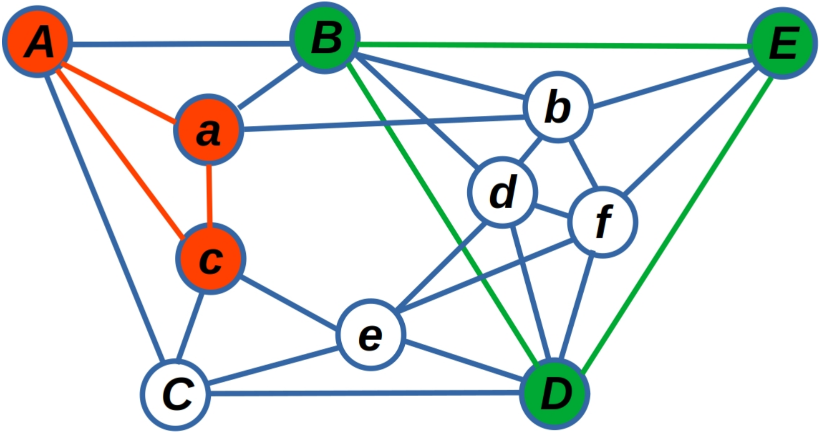
Maximum Independent Set



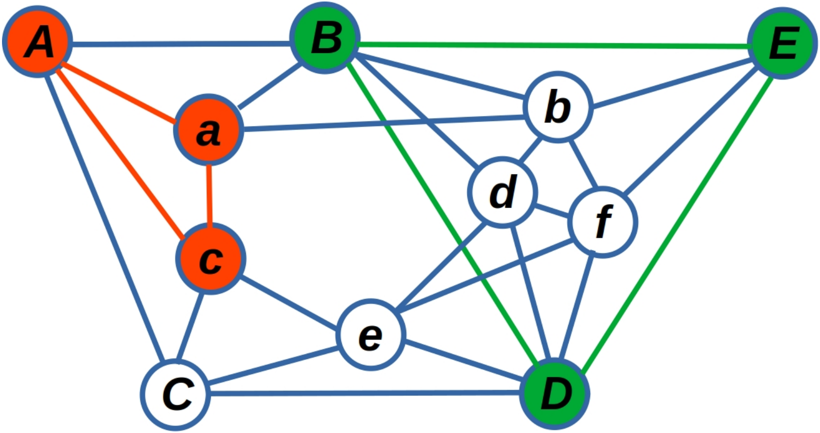
Clique



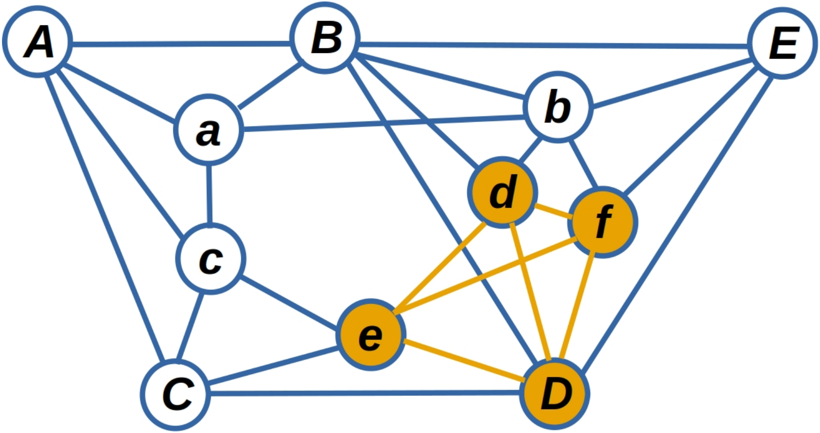
Clique



Clique: Is it maximal and maximum?



Maximum Clique



Few Bounds

Bounds for Chromatic Number[4,5]

Lower Bound: Based on Independence Number

Let $\alpha(G)$ be the independence number of graph G, and V is the vertex set, then

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$

Lower Bound: Based on Clique Number

Let $\omega(G)$ be the clique number of graph G, then

$$\chi(G) \geq \omega(G).$$

[4] Korman, S.M., 1979. The graph-colouring problem. Combinatorial optimization, pp.211-235.

[5] Lewis, R., 2015. A guide to graph colouring (Vol. 7). Berlin: Springer.

Few Other Bounds

Lower Bound: Based on number of vertices and edges

Let $G = (V, E)$ be the given graph, then

$$\chi(G) \geq \frac{|V|^2}{|V|^2 - 2|E|}.$$

Upper Bound: Based on number of vertices and edges

Let $G = (V, E)$ be the given graph, then

$$\chi(G) \leq 1 + \sqrt{\frac{2|E|(|V| - 1)}{|V|}}.$$

Few Other Bounds

Trivial Upper Bound: Based on maximum degree [6]

Let G be a graph and the maximum degree is Δ, then

$$\chi(G) \leq \Delta + 1.$$

More Refined Upper Bound: Based on maximum degree[7], [8]

Let G be a graph other than an odd-length cycle or a complete graph, and the maximum degree is Δ, then

$$\chi(G) \leq \Delta.$$

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[6] Ore, O., 1962. Theory of Graphs, vol. 38. In American Mathematical Society Colloquium Publications.

[7] Brooks, R.L., 1941, April. On colouring the nodes of a network. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 37, No. 2, pp. 194-197). Cambridge University Press.

[8] Lovász, L., 1975. Three short proofs in graph theory. Journal of Combinatorial Theory, Series B, 19(3), pp.269-271.

Few Other Bounds

Another Upper Bound: Based on degree [9]

Let $G = (V, E)$ be a graph. Let the vertices be labeled from 1 to $|V|$ as per non-increasing order according to degree. Let $d(i)$ be the degree of vertex i. Then,

$$\chi(G) \leq \max_{1 \leq i \leq |V|} \min[d(i) + 1, i].$$

[9] Welsh, D.J. and Powell, M.B., 1967. An upper bound for the chromatic number of a graph and its application to timetabling problems. The Computer Journal, 10(1), pp.85-86.

Mathematical Formulations

Formulation 1 [10]

$$min: \quad \sum_{k=1}^{n} y_k \tag{1}$$

$$s.t.: \quad \sum_{k=1}^{n} x_{ik} = 1 \qquad \forall v_i \in V \tag{2}$$

$$x_{ik} + x_{jk} \leq 1 \qquad \forall (v_i, v_j) \in E \tag{3}$$

$$y_k \geq x_{ik} \qquad \forall v_i \in V,\ k = 1, 2, \cdots, n \tag{4}$$

$$y_k, x_{ik} \in \{0, 1\} \qquad \forall v_i \in V,\ k = 1, 2, \cdots, n \tag{5}$$

- $y_k = 1$ if color k is used.
- $x_{ik} = 1$ if color k is assigned to vertex v_i.
- Polynomial number of variables and constraints.
- Optimal solution is the chromatic number.

[10] Pardalos, P.M., Mavridou, T. and Xue, J., 1998. The graph coloring problem: A bibliographic survey. Handbook of Combinatorial Optimization: Volume1–3, pp.1077-1141.

Formulation 2 [Pardalos et al.:1998]

$$
\begin{aligned}
min: \quad & \gamma && && (6)\\
s.t.: \quad & x_i \leq \gamma && \forall v_i \in V && (7)\\
& x_i - x_j - 1 \geq -n\delta_{ij} && \forall (v_i, v_j) \in E && (8)\\
& x_j - x_i - 1 \geq -n(1 - \delta_{ij}) && \forall (v_i, v_j) \in E && (9)\\
& x_i \in Z^+ && \forall v_i \in V && (10)\\
& \delta_{ij} \in \{0, 1\} && \forall v_i, v_j \in V && (11)
\end{aligned}
$$

- x_i is the color assigned to vertex v_i.
- No feasible value of δ_{ik} will satisfy constraint 8 and 9 if $x_i = x_j$ for $(v_i, v_j) \in E$.
- Polynomial number of variables and constraints
- Optimal solution is the chromatic number

$$min: \quad \sum_{j=1}^{t} s_j \tag{12}$$

$$s.t.: \quad \sum_{j=1}^{t} e_{ij} s_j = 1 \qquad i = 1, 2, \cdots, n \tag{13}$$

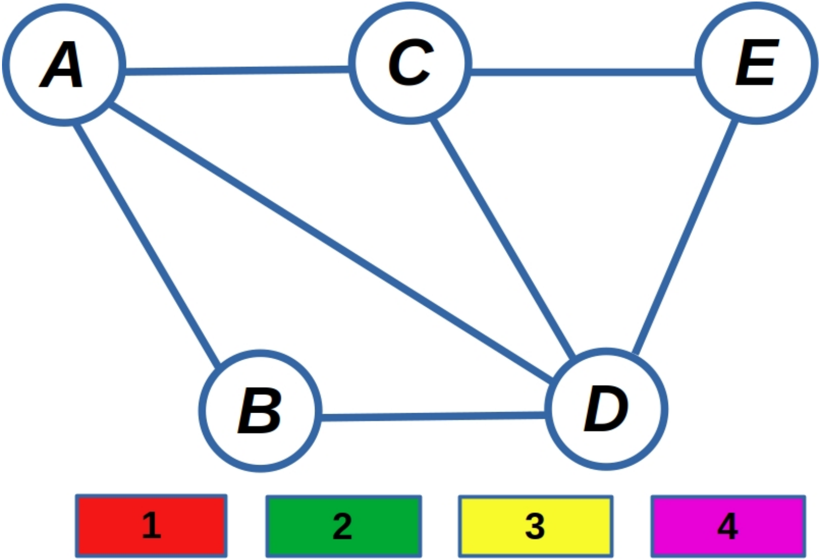
$$s_j \in \{0, 1\} \qquad j = 1, 2, \cdots, t \tag{14}$$

- $S_1, S_2, \cdots, S_t$ be all possible independent sets.
- $s_j = 1$ if S_j is chosen color class
- $e_{ij} = 1$ if vertex $v_i \in S_j$.
- Exponential number of variables.
- Convenient for Column Generation approach.[11]
- Optimal solution is the chromatic number.

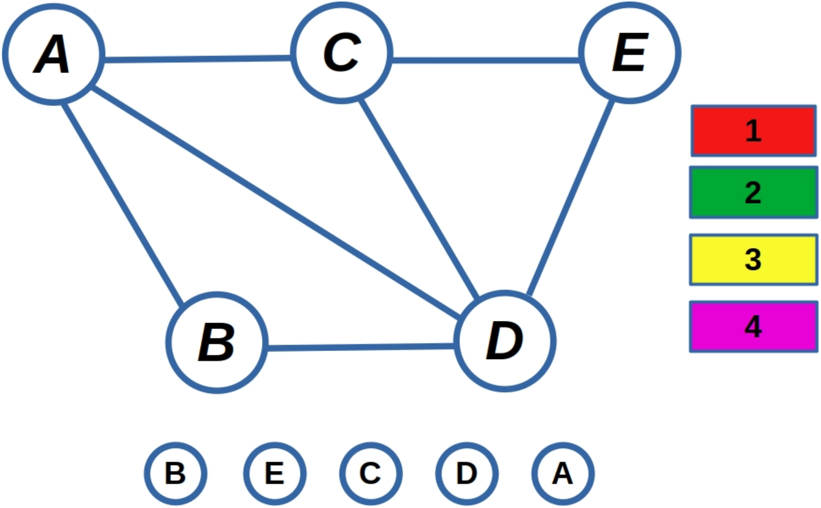
[11] Mehrotra, A. and Trick, M.A., 1996. A column generation approach for graph coloring. Informs Journal on Computing, 8(4), pp.344-354.

Algorithms ad Heuristics

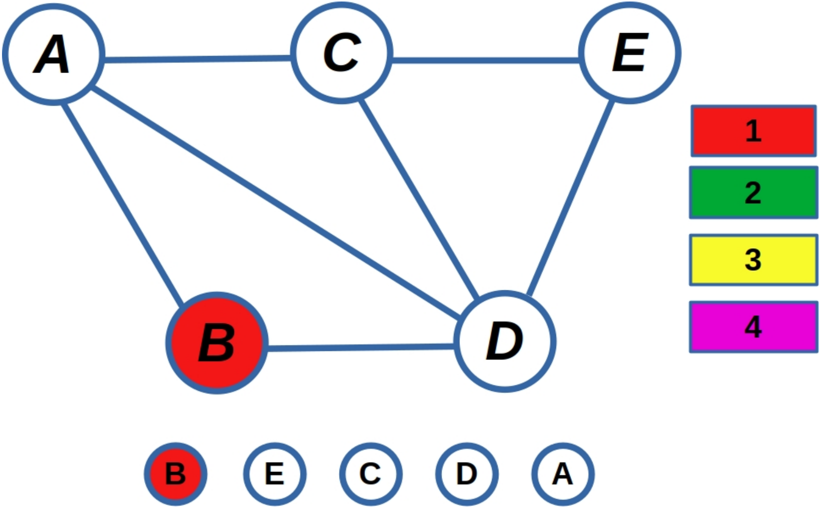
A Simple Greedy Algorithm



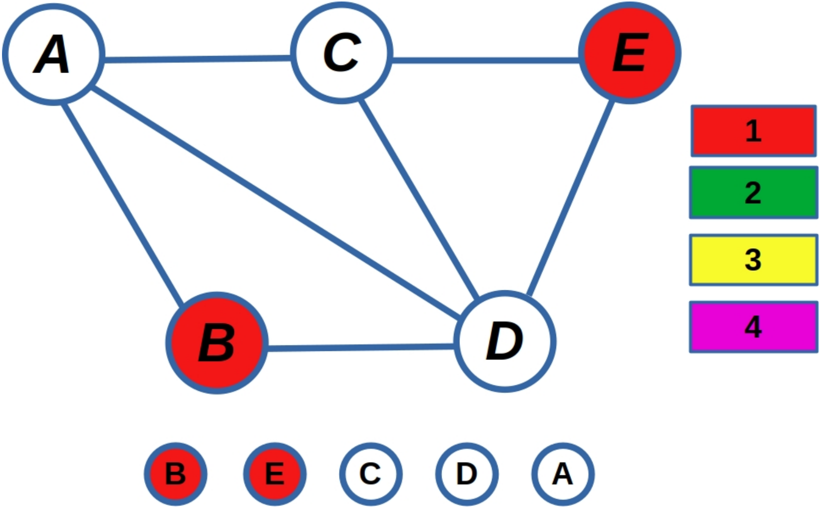
Greedy Algorithm: Step1 - Consider an Ordering of Vertices



Greedy Algorithm: Step2 (Loop)- Assign least numbered color to the next uncolored vertex as per valid coloring



Greedy Algorithm: Step2 (Loop)- Assign least numbered color to the next uncolored vertex as per valid coloring



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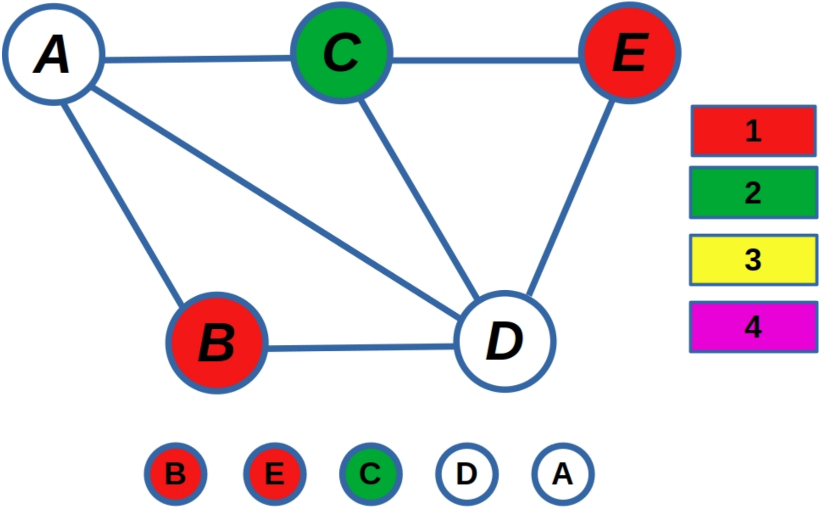
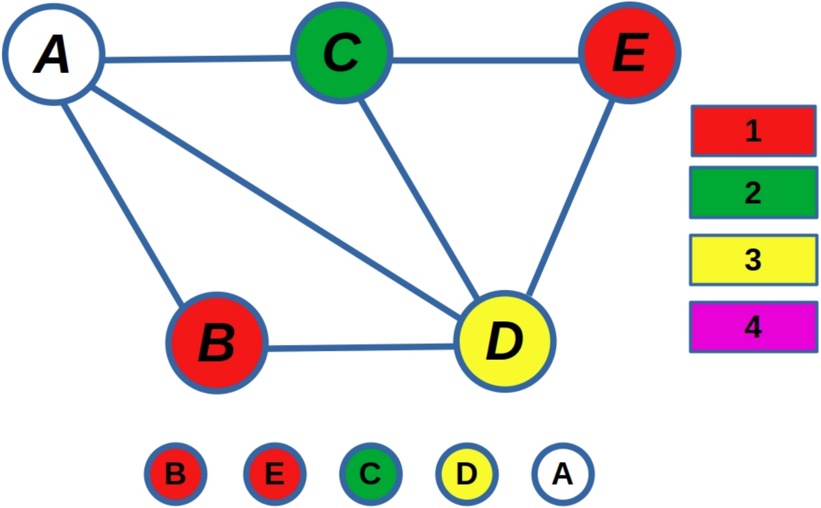


Greedy Algorithm: Step2 (Loop)- Assign least numbered color to the next uncolored vertex as per valid coloring



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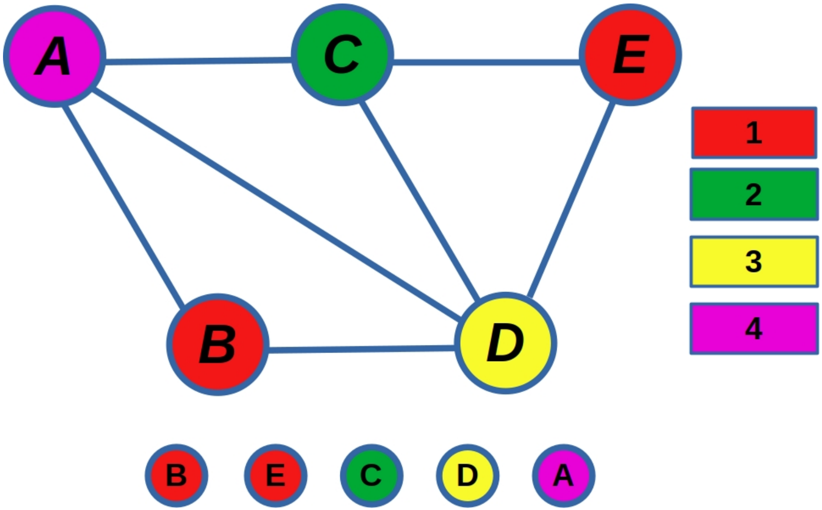
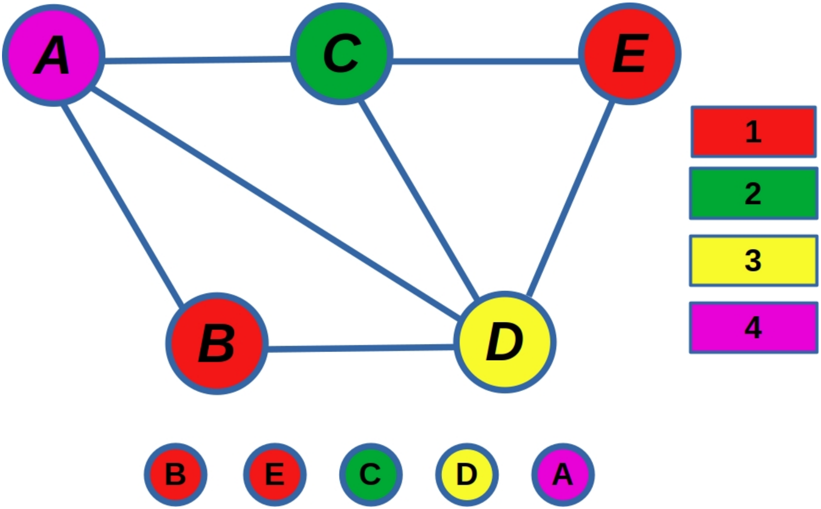
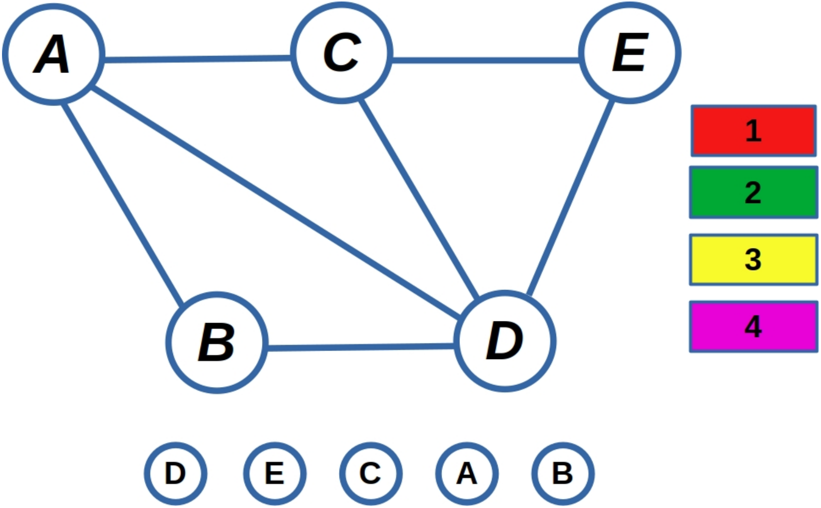


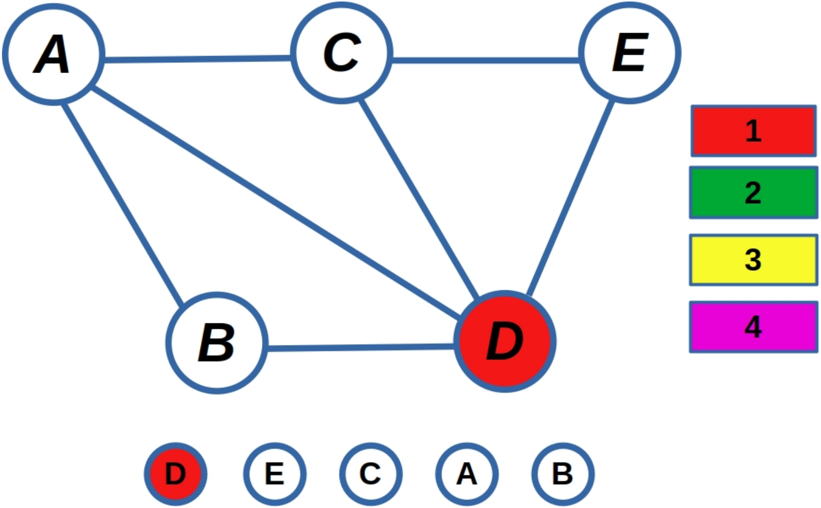
Is it optimal?



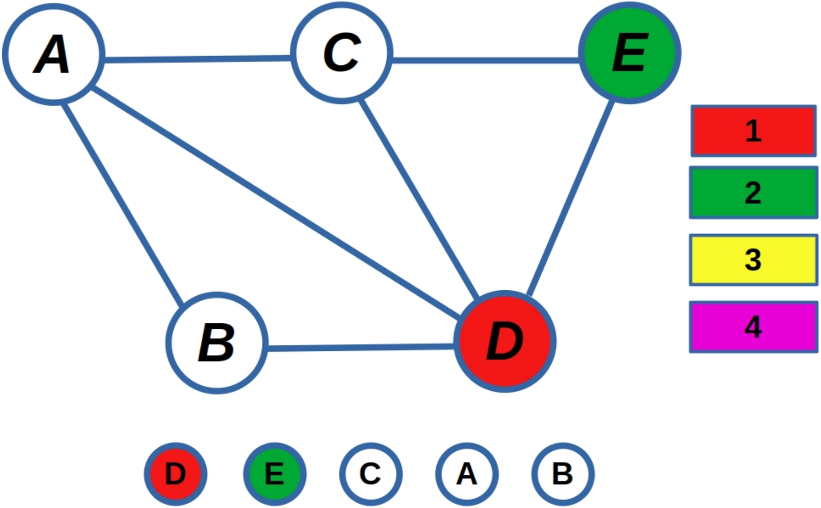
Consider another Ordering of Vertices



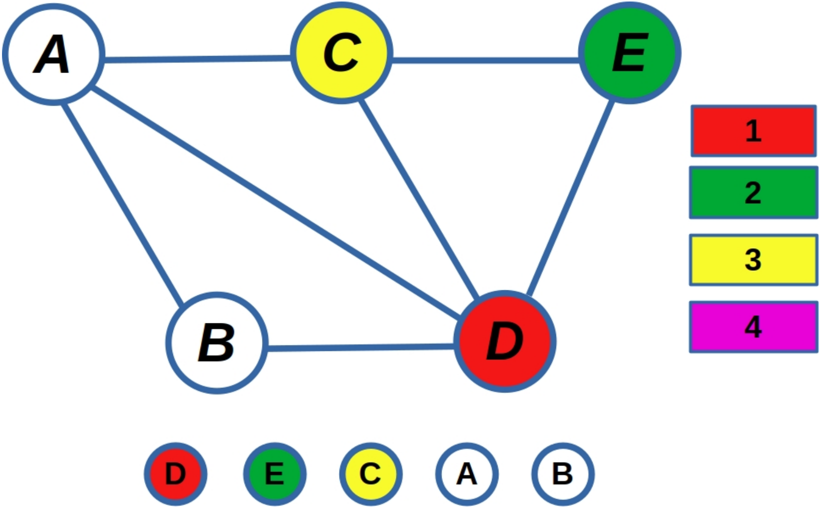
Assign least numbered color to the next uncolored vertex as per valid coloring



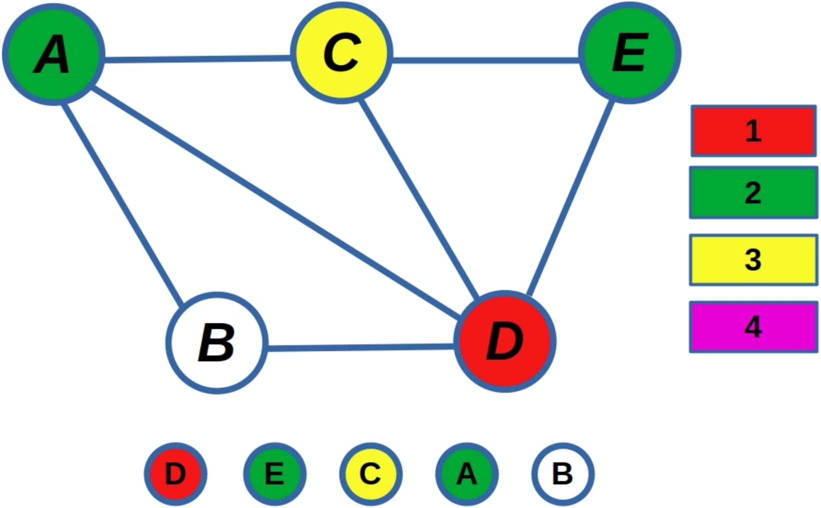
Assign least numbered color to the next uncolored vertex as per valid coloring



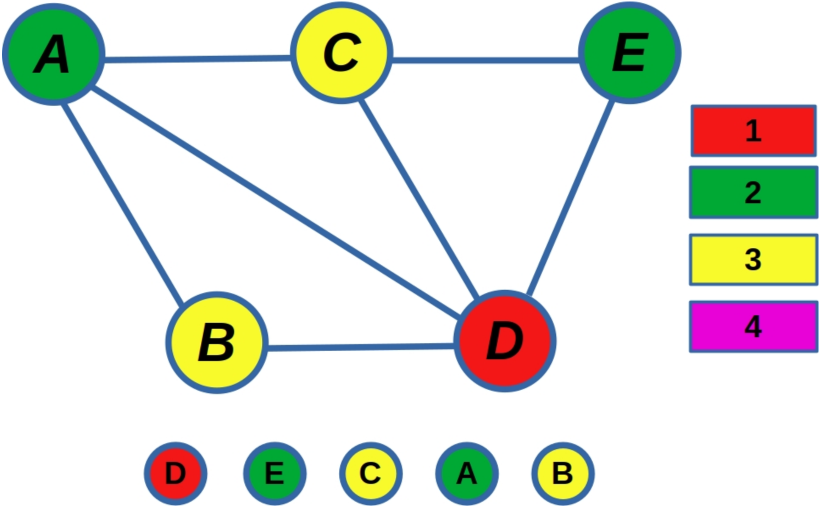
Assign least numbered color to the next uncolored vertex as per valid coloring



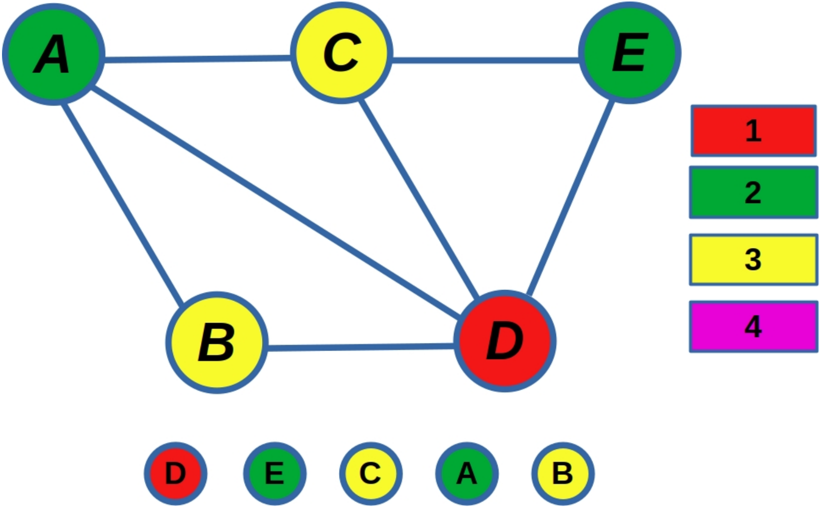
Assign least numbered color to the next uncolored vertex as per valid coloring



Assign least numbered color to the next uncolored vertex as per valid coloring



Is it optimal now? Conclusion?



Facts: Greedy Algorithms

- There exists an ordering of vertices for which greedy algorithms gives the exact solution.
- Optimal for interval graphs when intervals are kept in non-decreasing order according to the arrival time.
- Arbitrary ordering can be bad. Over Crown graphs, greedy coloring may require $\frac{n}{2}$ colors when it can be colored with 2 colors.
- **Grundy number:** Maximum number of colors that may be required by a greedy coloring algorithm.

Ordering Schemes: Decide before starting coloring

- Largest First[12]:
  - Decreasing order of degrees.
  - Uses at most one more than the graph's maximum degree colors.
- Smallest Last[13]:
  - The minimum degree vertex is ordered last.
  - In each iteration i, the minimum degree vertex is deleted and kept as the i-th vertex from the last.

[12] Welsh, D.J. and Powell, M.B., 1967. An upper bound for the chromatic number of a graph and its application to timetabling problems. The Computer Journal, 10(1), pp.85-86.

[13] Matula, D.W., Marble, G. and Isaacson, J.D., 1972. Graph coloring algorithms. In Graph theory and computing (pp. 109-122). Academic Press.

Ordering Schemes: Decide during coloring

- DSatur[14]:
  - Color the maximum saturation degree vertex next.
  - $O(|V|^2)$ running time. $O(|E|(log|V|))$ when implemented using binary heap.
  - Optimal Coloring for Bipartite graphs, Cycle graphs, and Wheel graphs [Lewis:2015].
- Recursive Largest Fit[15]:
  - Generates color classes one by one. Chooses the next vertex in a color class with maximum degree among the potential vertices that can belong in that color class)

[14] Brélaz, D., 1979. New methods to color the vertices of a graph. Communications of the ACM, 22(4), pp.251-256.

[15] Leighton, F.T., 1979. A graph coloring algorithm for large scheduling problems. Journal of research of the national bureau of standards, 84(6), p.489.

Other Type of Algorithms for Graph Coloring

- Brute-Force
- Backtracking
- Integer Programming Based
- Column Generation
- Genetic
- Tabu search
- Simulated annealing
- Ant Colony Optimization
- Hill Climbing

- Determining whether a graph is k-colorable is NP-complete.[16]
- 2-COL is polynomial time solvable.
- SAT $\leq_P$ 3-CNF-SAT $\leq_P$ 3-COL $\leq_P$ k-COL ($k \geq 3$).
- Unless P=NP, for $f < 2$, there does not exists any f-factor approximation algorithm for graph coloring.[17]
- Approximating $\chi(G)$ within n^{ϵ}, $\epsilon > 0$ is NP-hard. [18]

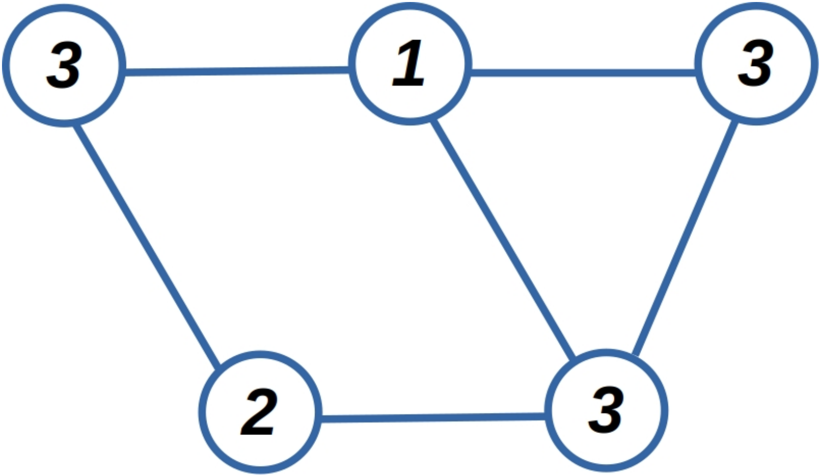
[16] Karp, R.M., 1972. Complexity of computer computations. Reducibility among combinatorial problems, 23(1), pp.85-103.

[17] Garey, M.R. and Johnson, D.S., 1976. The complexity of near-optimal graph coloring. Journal of the ACM (JACM), 23(1), pp.43-49.

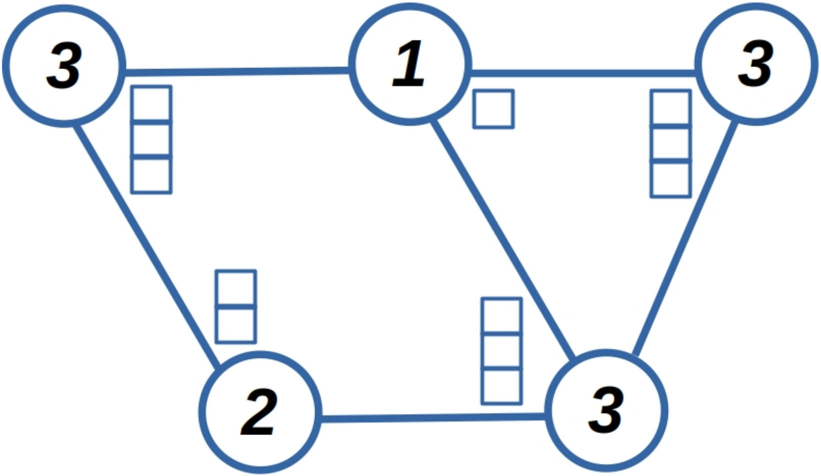
[18] Khanna, S., Linial, N. and Safra, S., 2000. On the hardness of approximating the chromatic number. Combinatorica, 20(3), pp.393-415.

Generalizations and Variants

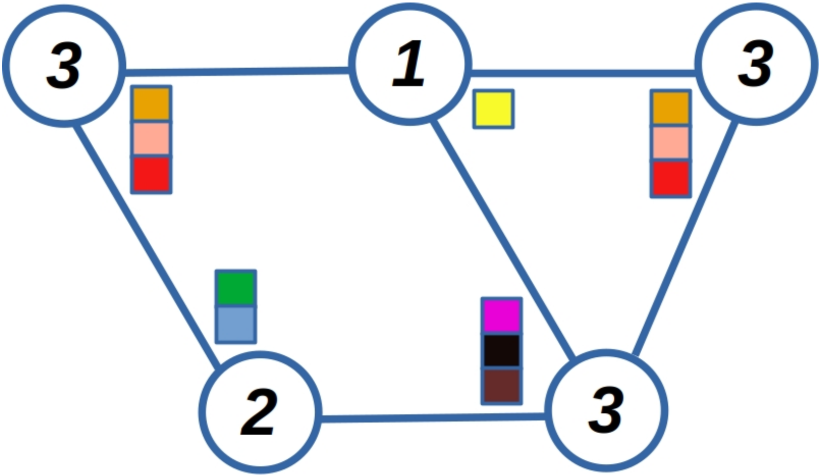
Weighted Coloring



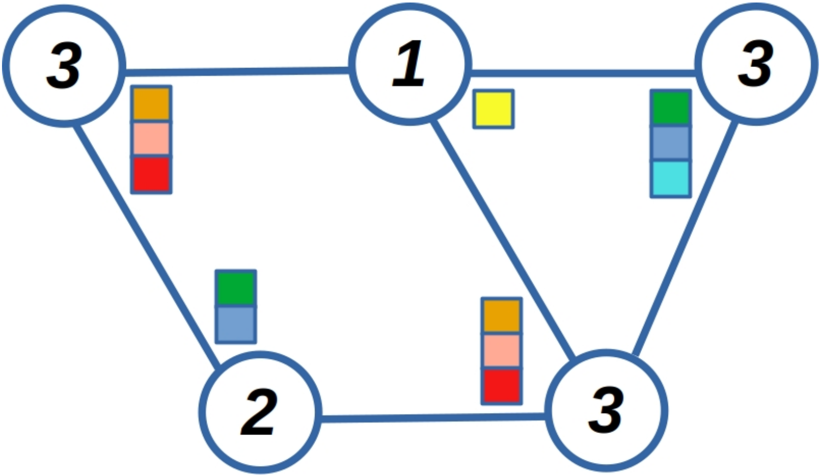
Weighted Coloring



Weighted Coloring: Is it optimal?



Weighted Coloring: Is it optimal?



List Coloring

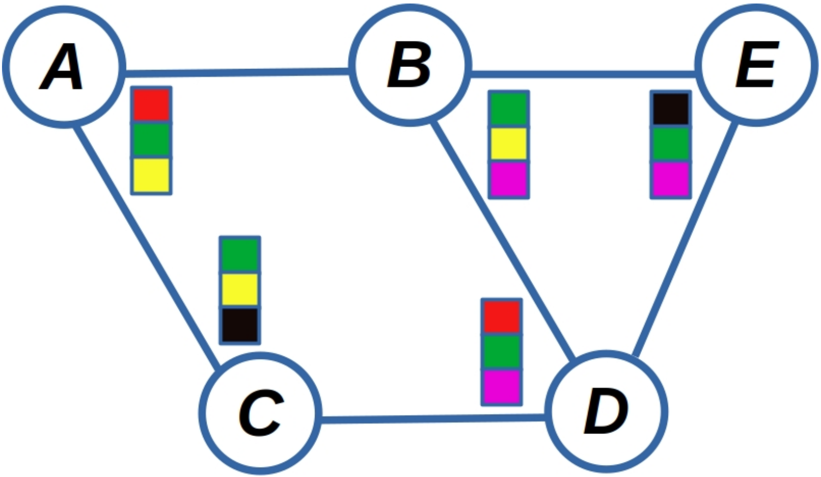
- Introduced by Vizing[19] and Erdos et al. [20].
- Application in channel assignment problem: Lists of acceptable channels are specified by transmitters.
- *k*-**choosable**: Possible to list color no matter how the list of k colors are choosen at each node.
- **Choosability**(List Colorability/ Chromatic Number): $ch(G)$:- Least k such that graph G is k-choosable.
- Trivial: $ch(G) \geq \chi(G)$
- List coloring is NP-complete even for interval graphs.[21]

[19] Vizing, V.G., 2000. Vertex colorings with given colors, Metody Diskret. Analiz. 29 (1976) 3–10.

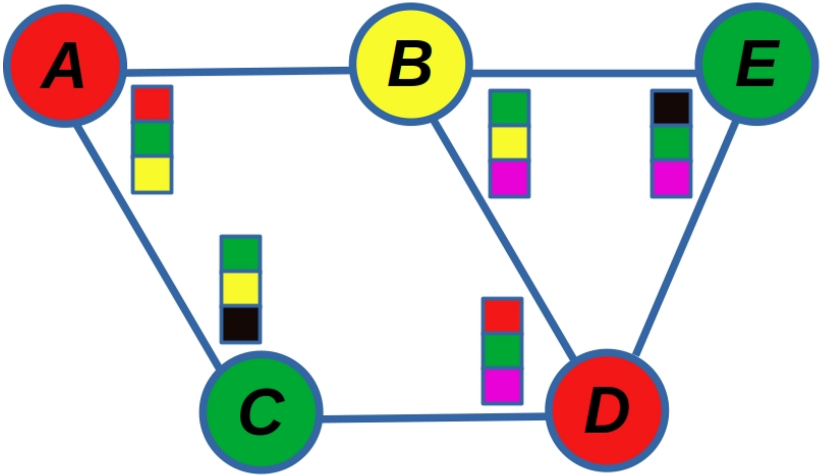
[20] Erdos, P., Rubin, A.L. and Taylor, H., 1979, September. Choosability in graphs. In Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium (Vol. 26, pp. 125-157).

[21] Biró, M., Hujter, M. and Tuza, Z., 1992. Precoloring extension. I. Interval graphs. Discrete Mathematics, 100(1-3), pp.267-279.

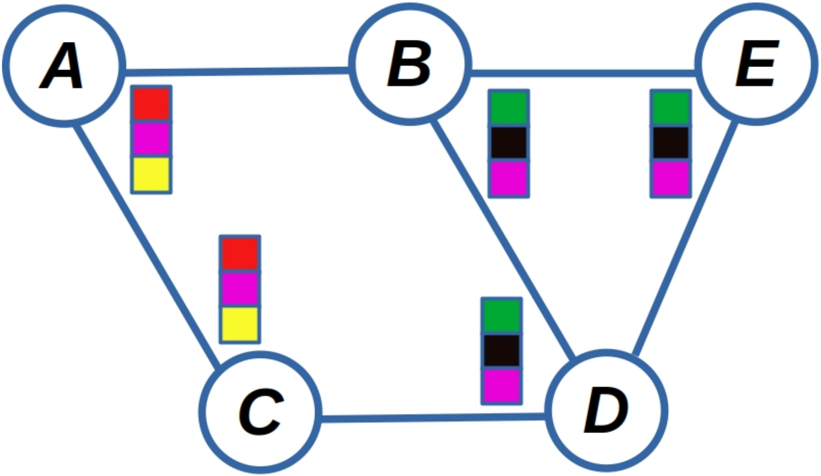
List Coloring



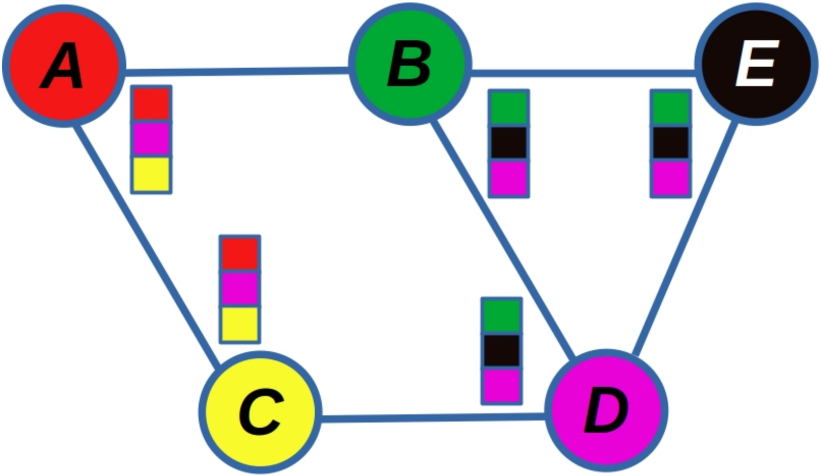
List Coloring



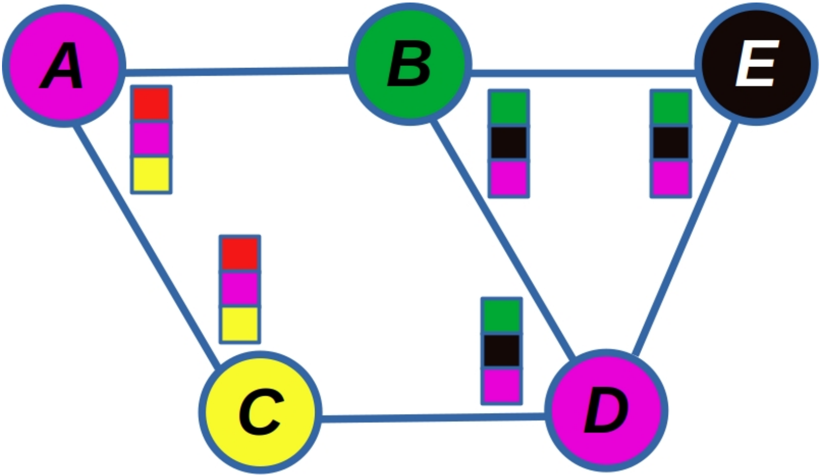
List Coloring



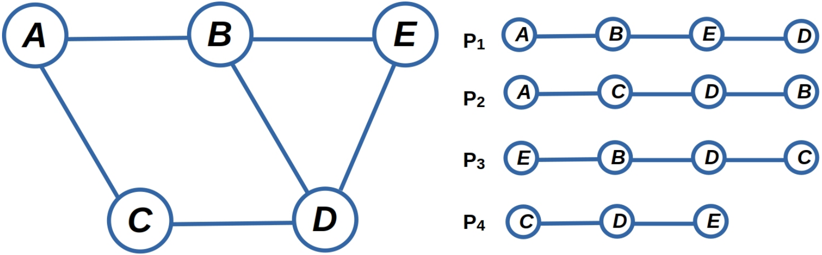
List Coloring:



List Coloring:

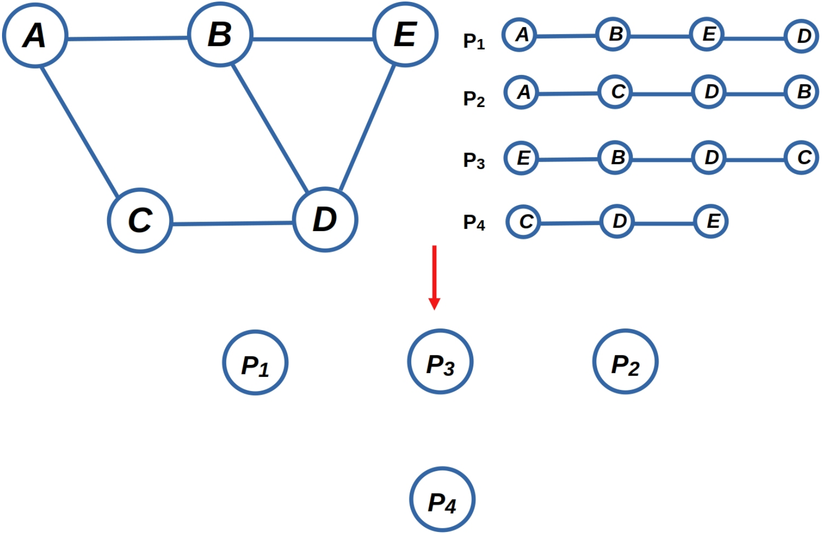


Path Coloring[22]



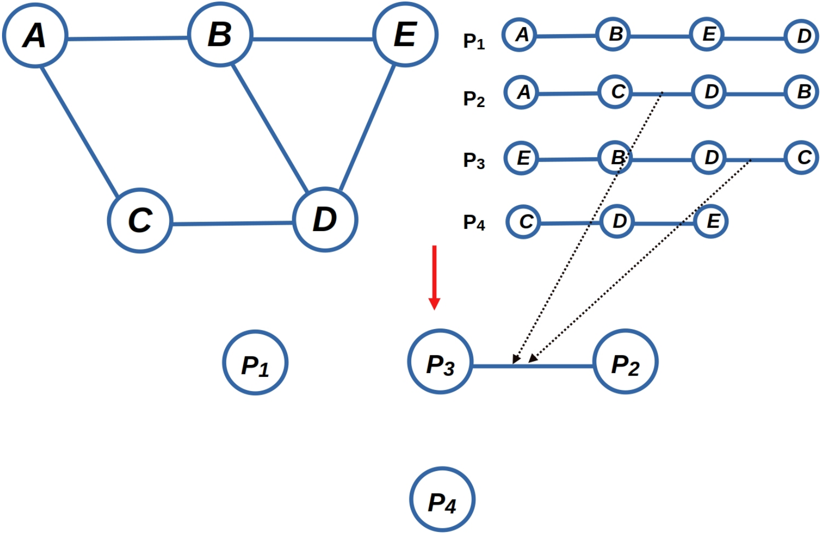
[22] Erlebach, T. and Jansen, K., 2001. The complexity of path coloring and call scheduling. Theoretical Computer Science, 255(1-2), pp.33-50.

Path Coloring[22]



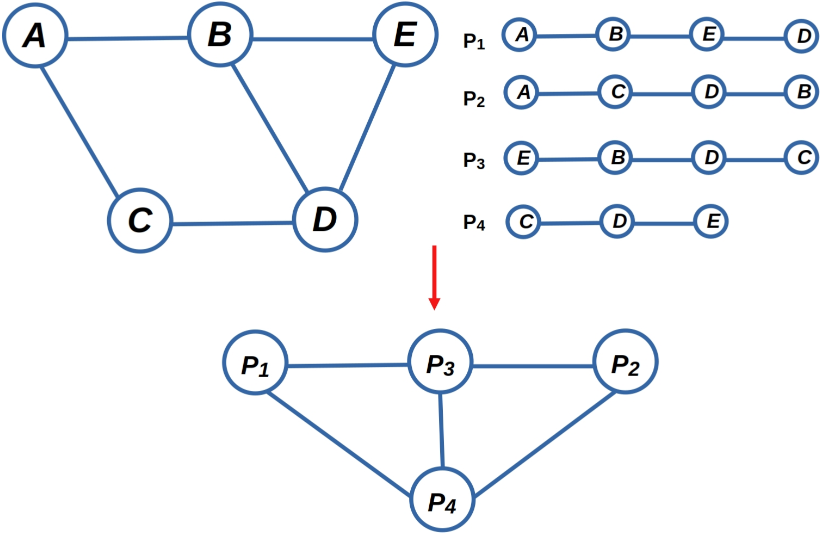
[22] Erlebach, T. and Jansen, K., 2001. The complexity of path coloring and call scheduling. Theoretical Computer Science, 255(1-2), pp.33-50.

Path Coloring[22]



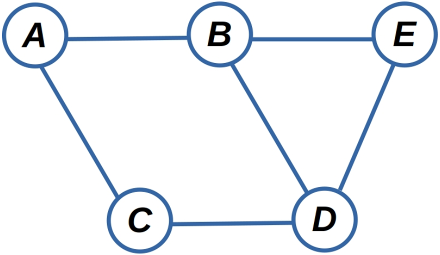
[22] Erlebach, T. and Jansen, K., 2001. The complexity of path coloring and call scheduling. Theoretical Computer Science, 255(1-2), pp.33-50.

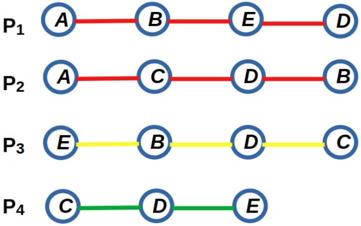
Path Coloring[22]

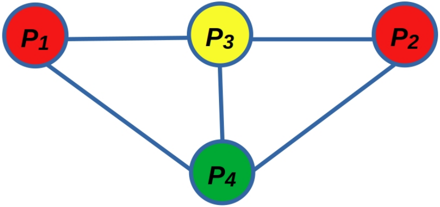


22 Erlebach, T. and Jansen, K., 2001. The complexity of path coloring and call scheduling. Theoretical Computer Science, 255(1-2), pp.33-50.

Path Coloring[22]

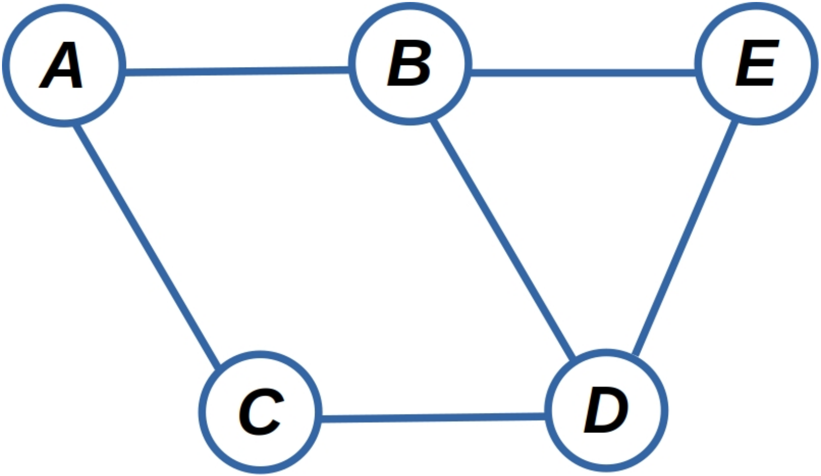




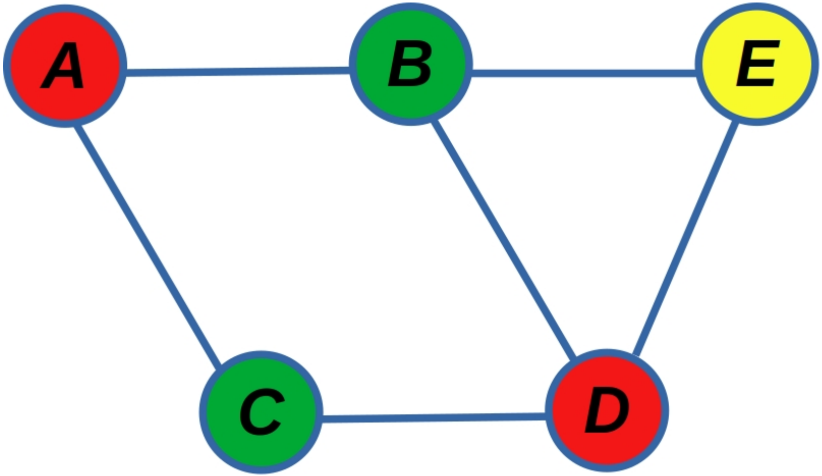


[22] Erlebach, T. and Jansen, K., 2001. The complexity of path coloring and call scheduling. Theoretical Computer Science, 255(1-2), pp.33-50.

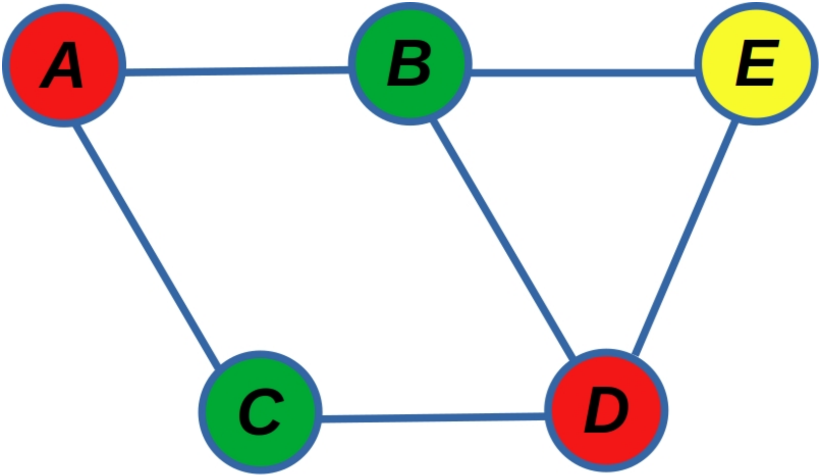
Acyclic Coloring



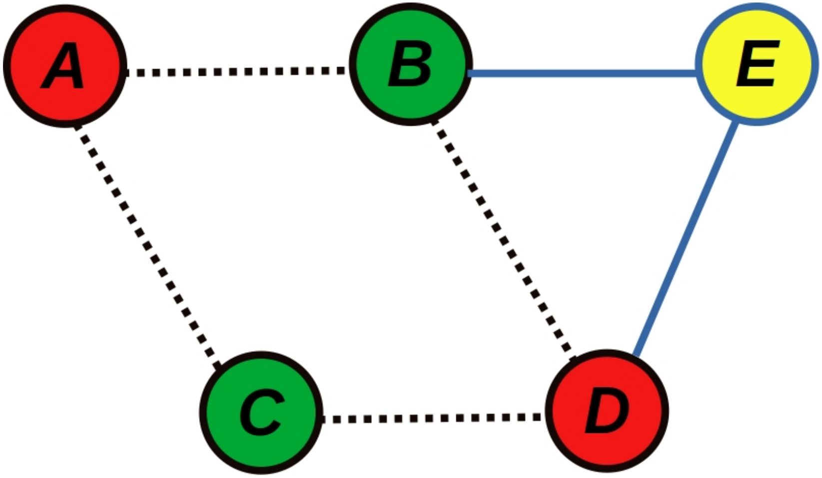
Acyclic Coloring



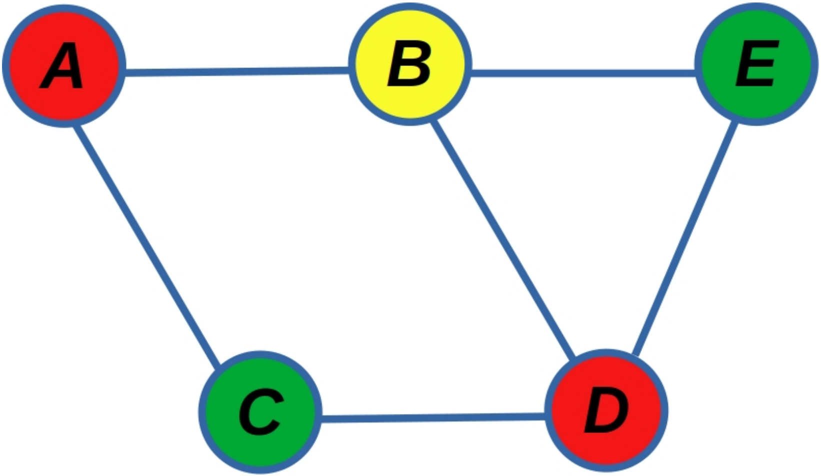
Acyclic Coloring: Is it Valid?



Acyclic Coloring: Is it Valid?



A Valid Acyclic Coloring



Acyclic and Star Coloring

- Introduced by Grünbaum[23]
- **Star Chromatic Number**: $\chi_S(G)$ - Minumum number of colors such that graph G has star coloring.
- **Acyclic Chromatic Number**: $A(G)$ - Minumum number of colors such that graph G has acyclic coloring.
- Let P be a planar graph then $7 \leq \chi_S(P) \leq 80$ [Fertin et al.][24]
- $\chi_S(K_{m,n}) = \min[m, n] + 1$ [Fertin et al.: 2001]
- Let G be a planar bipartite graph. It is NP-complete to determine whether $\chi_S(G) \leq 3$. [Albertson et al.][25]
- Optimal acyclic and star coloring of cographs in linear time.[26]

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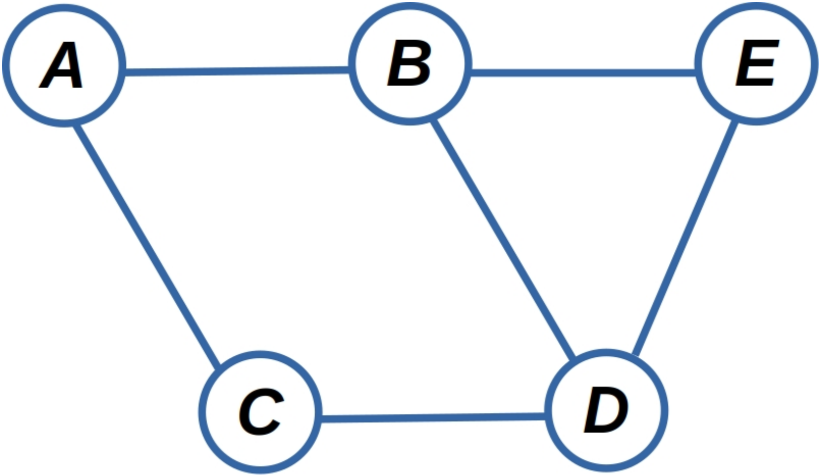
[23] Grünbaum, B., 1973. Acyclic colorings of planar graphs. Israel journal of mathematics, 14(4), pp.390-408.

[24] Fertin, G., Raspaud, A. and Reed, B., 2001. On star coloring of graphs. In Graph-Theoretic Concepts in Computer Science: 27th International Workshop, WG 2001 Boltenhagen, Germany, June 14–16, 2001 Proceedings 27 (pp. 140-153). Springer Berlin Heidelberg.

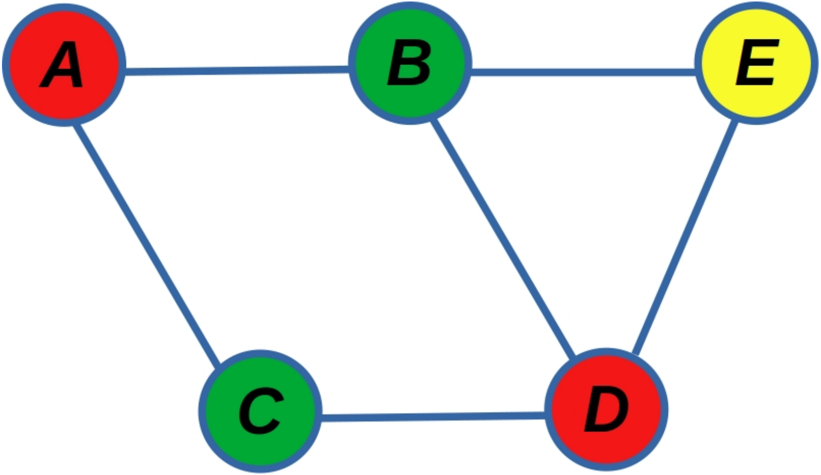
[25] Albertson, M.O., Chappell, G.G., Kierstead, H.A., Kündgen, A. and Ramamurthi, R., 2004. Coloring with no 2-Colored P_4's. the electronic journal of combinatorics, pp.R26-R26.

[26] Lyons, A., 2011. Acyclic and star colorings of cographs. Discrete applied mathematics, 159(16), pp.1842-1850.

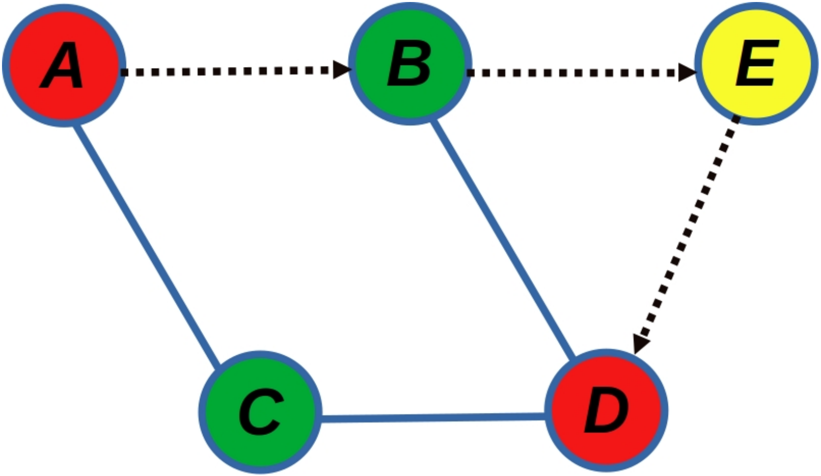
Star Coloring



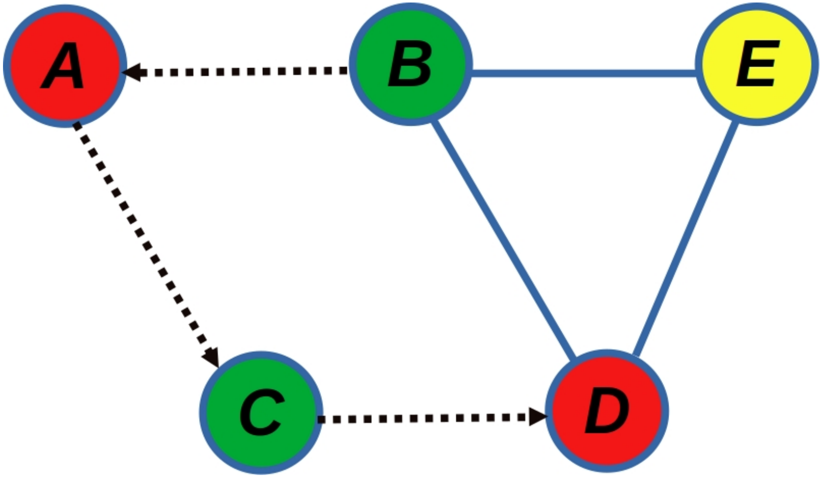
Star Coloring



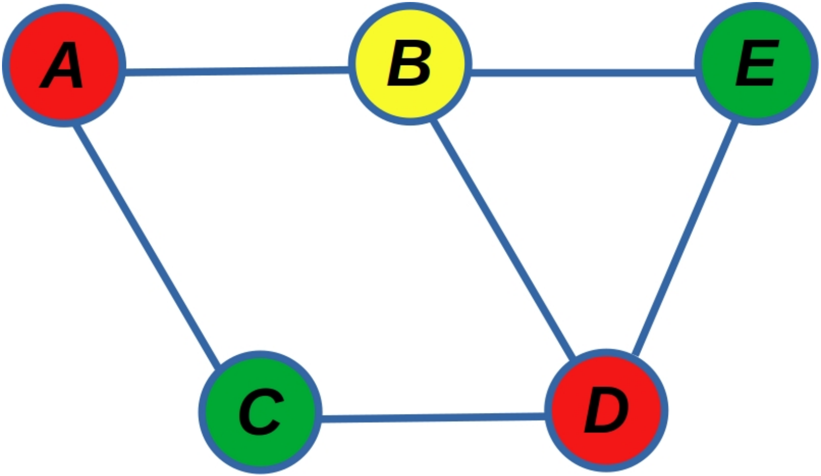
Star Coloring: Is it Valid?



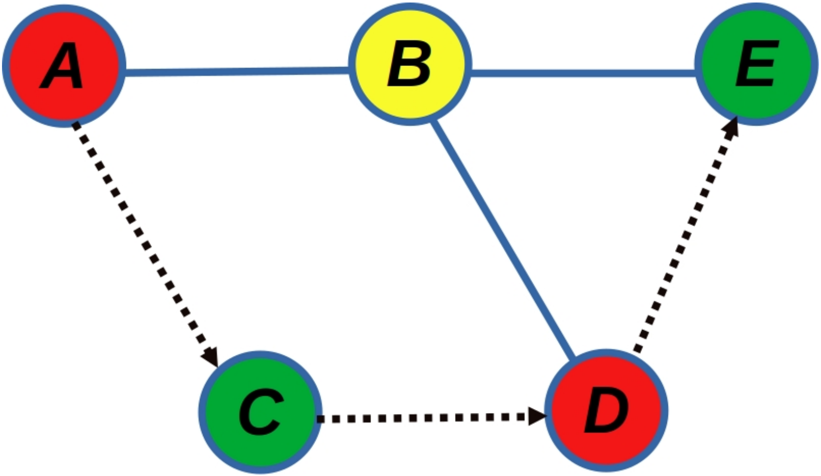
Star Coloring: Is it Valid?



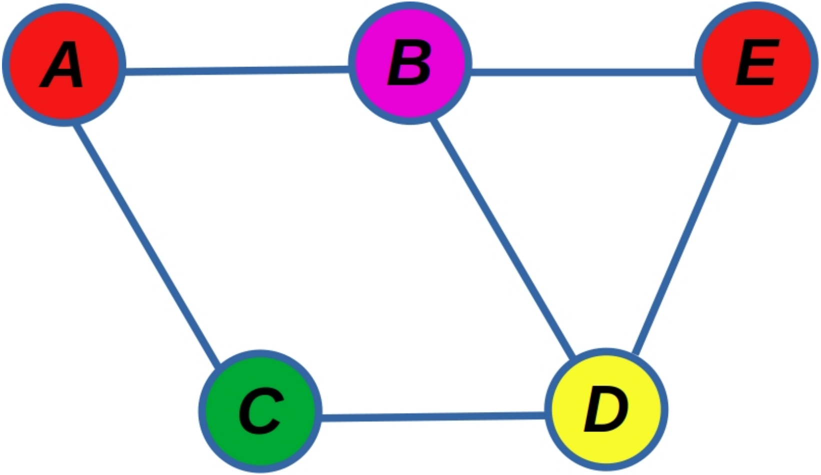
Star Coloring: Is it Valid?



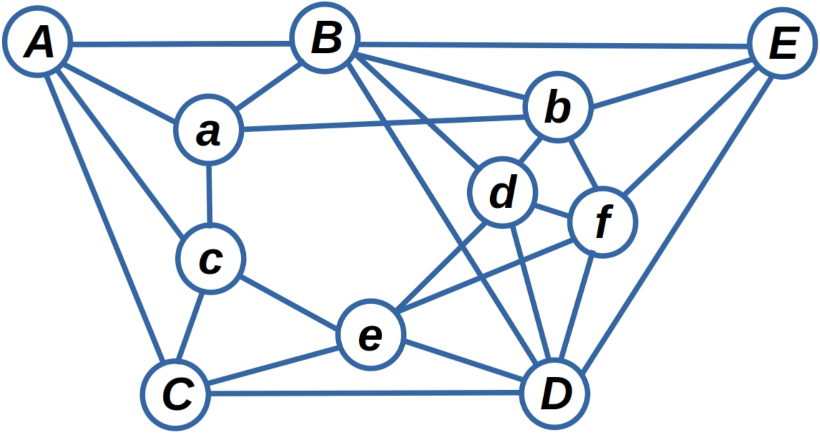
Star Coloring: Is it Valid?



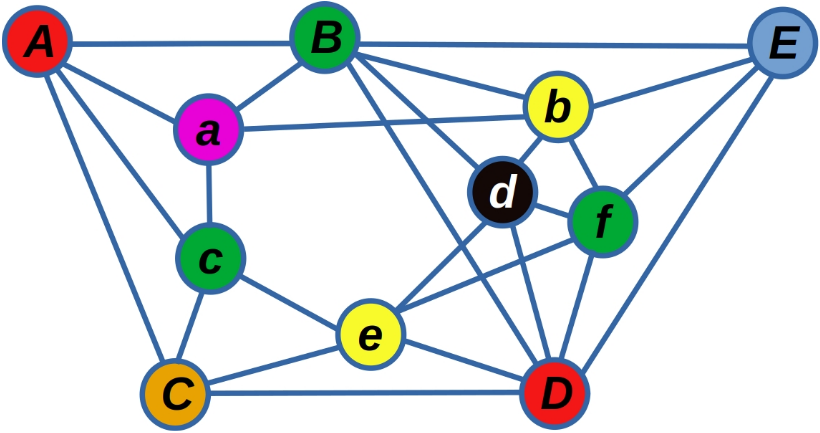
A Valid Star Coloring



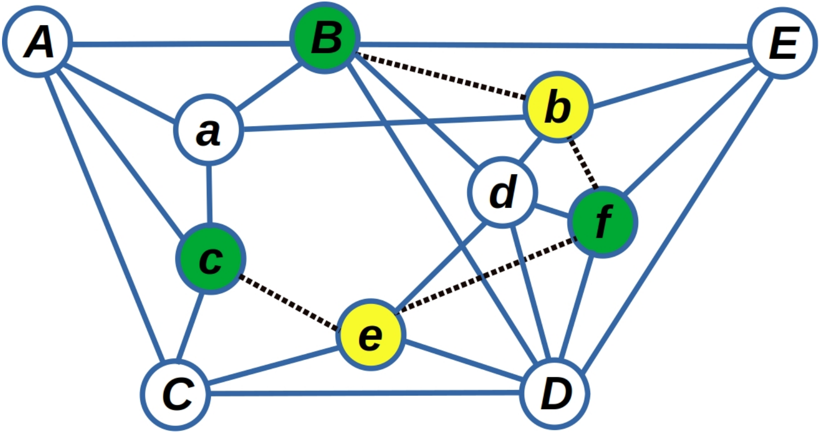
Star Coloring : Another Example



Star Coloring : Another Example



Star Coloring: how to validate?



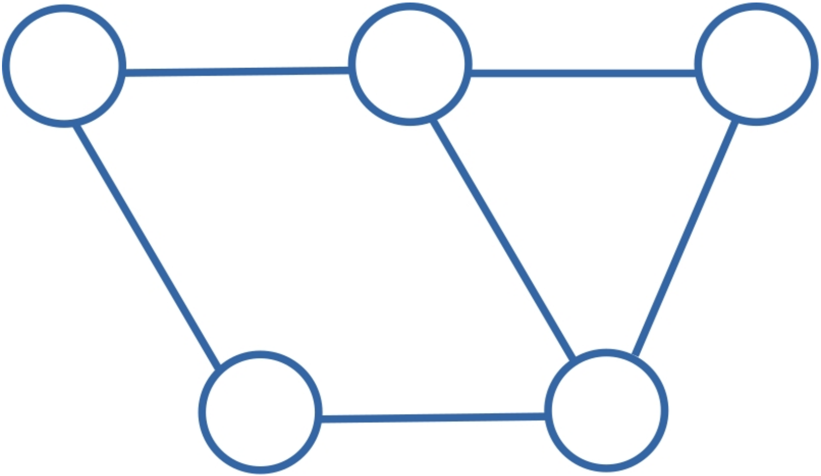
T-Coloring

- Introduced by Hale[27]
- Application in frequency assignment problem: The set T denotes separation not allowed between pair of interfering transmitters.
- **T-Chromatic Number**: $\chi_T(G)$:- Least number of colors such that graph G has T-coloring.
- **Span of a T-coloring**: Maximum difference between any pair of positive integers used for T-coloring.
- **T-span**: $sp_T(G)$:- minimum span of all possible T-colorings of a graph G.
- T-coloring is same as proper vertex coloring when $T = \{0\}$ and $sp_T(G) = \chi(G) - 1$.
- Finding $sp_T(G)$ is NP-complete. [28]

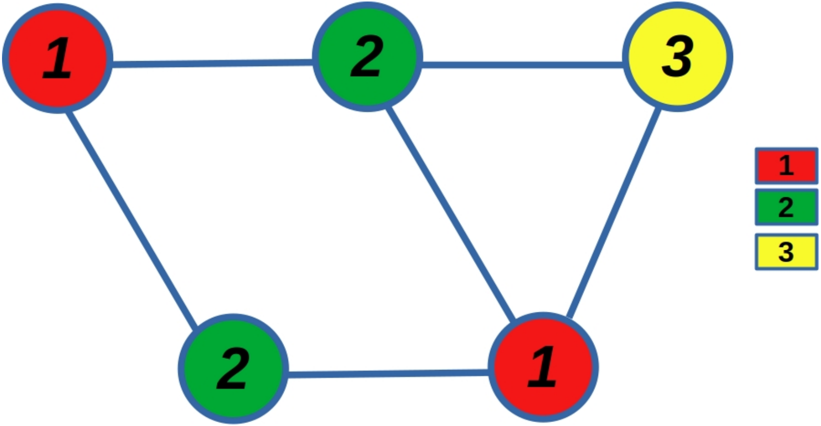
[27] Hale, W.K., 1980. Frequency assignment: Theory and applications. Proceedings of the IEEE, 68(12).

[28] Cozzens, M.B. and Wang, D.I., 1984. The general channel assignment problem. Congressus Numerantium, 41.

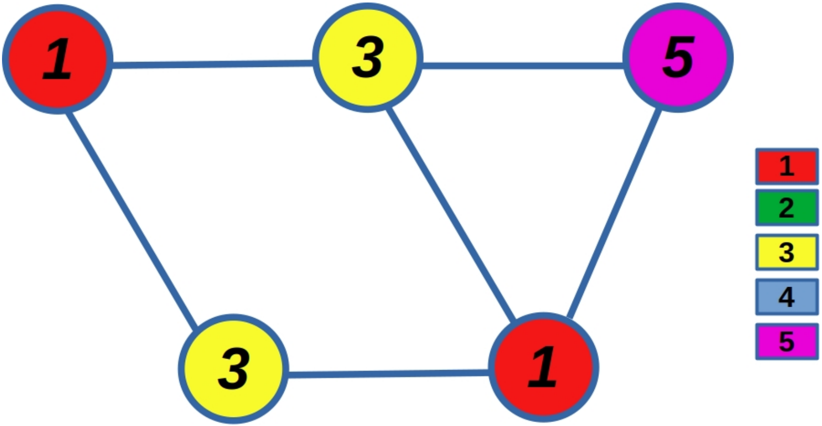
T-Coloring



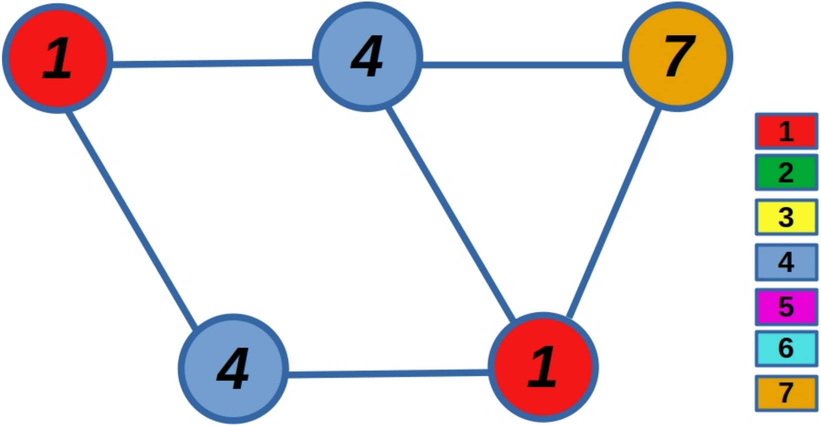
T-Coloring: $T = \{0\}$, span=2



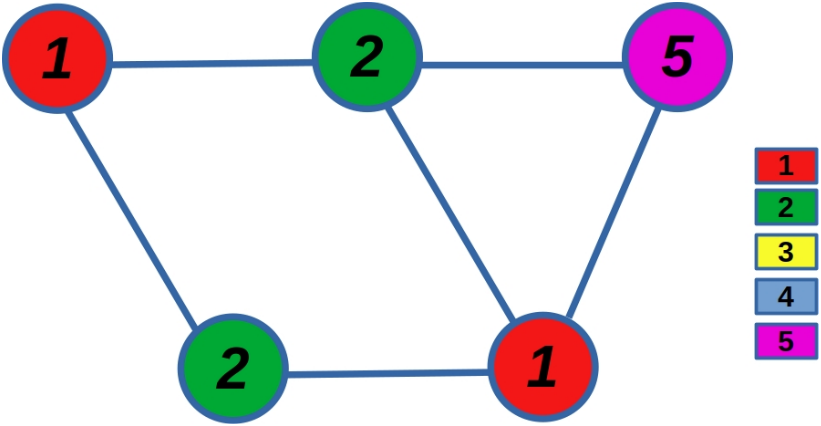
T-Coloring: $T = \{0, 1\}$, span=4



T-Coloring: $T = \{0, 1, 2\}$, span=6



T-Coloring: $T = \{0, 2\}$, span=4



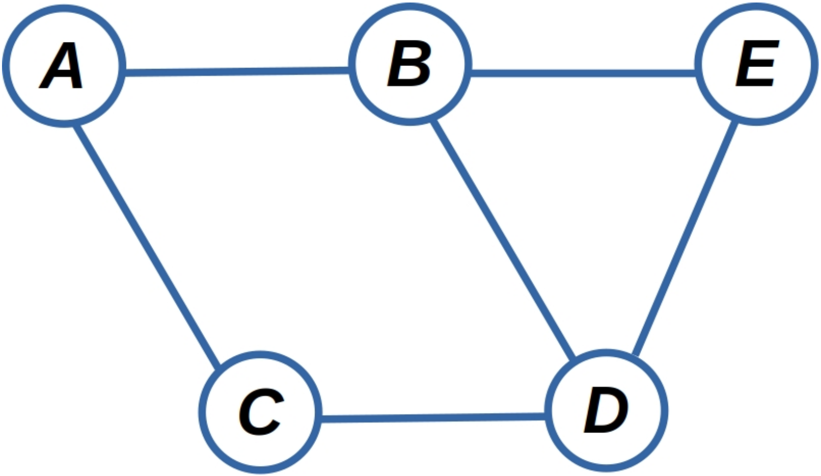
Equitable Coloring

- Application in job scheduling (of uniform sizes) on homogeneous machines where the objective is minimize the number of machines while ensuring almost equal number of jobs (equal load) at machines.
- **Equitable Chromatic Number**: $\chi_{=}(G)$:- Least number of colors such that graph G has equitable coloring.
- $\chi_{=}(G) \leq \Delta + 1$ where Δ is the maximum degree. [29]
- Equitable $(\Delta + 1)$-coloring for a graph $G = (V, E)$ with maximum degree Δ can be determined in $O(\Delta|V|^2)$.[30]
- Equitable coloring is NP-complete.

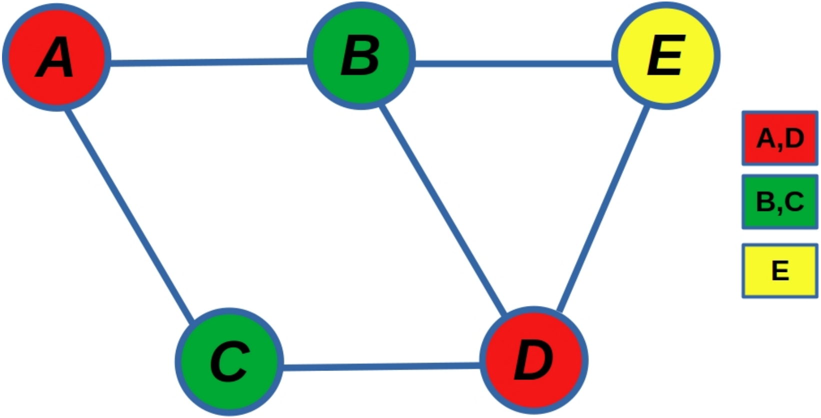
[29] Hajnal, A. and Szemerédi, E., 1970. Proof of a conjecture of P. Erdos. Combinatorial theory and its applications, 2, pp.601-623.

[30] Kierstead, H.A., Kostochka, A.V., Mydlarz, M. and Szemerédi, E., 2010. A fast algorithm for equitable coloring. Combinatorica, 30(2), pp.217-224.

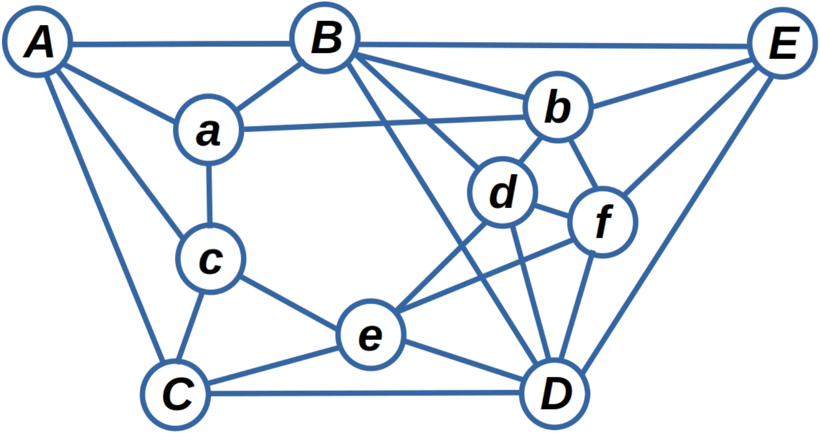
Equitable Coloring



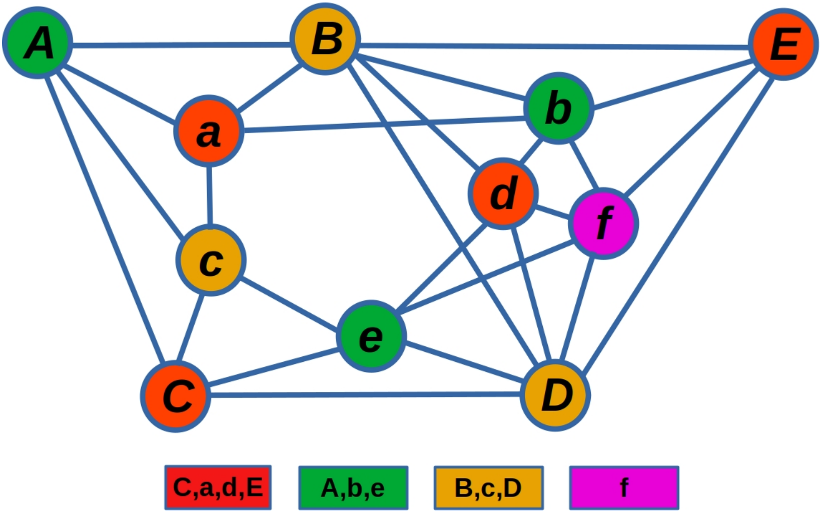
Equitable Coloring



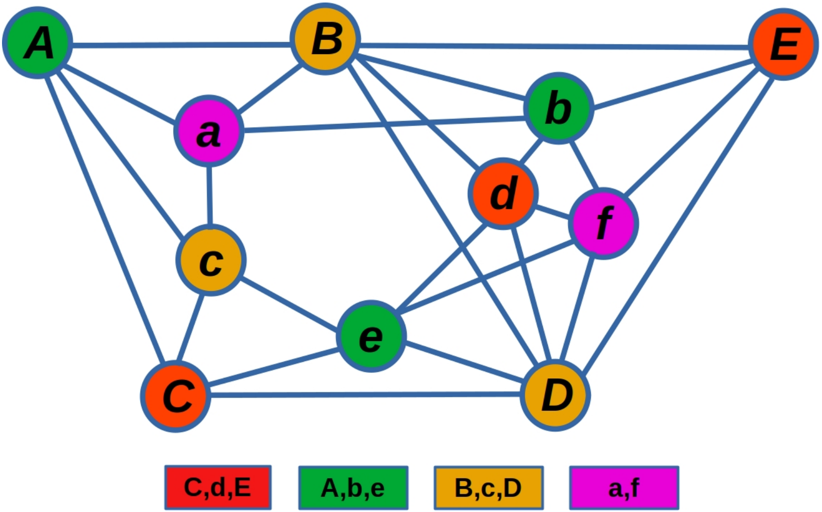
Equitable Coloring: Another Example



Equitable Coloring: Is it valid?



Equitable Coloring: Is it valid now? Is it optimal?

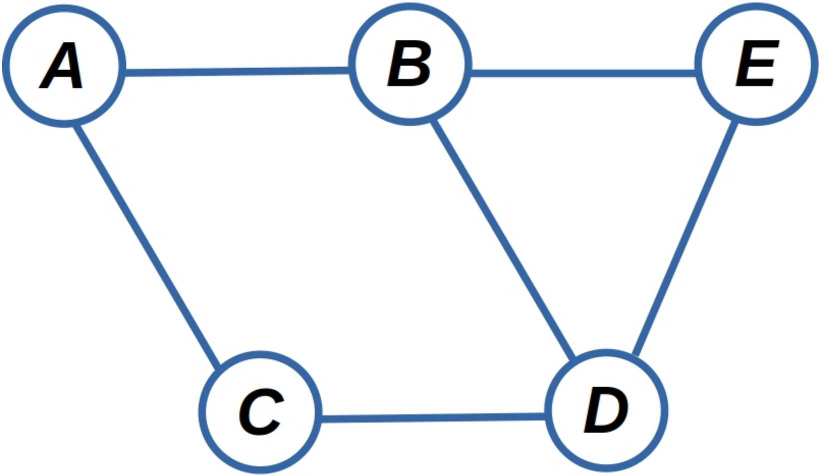


Improper Coloring[31]

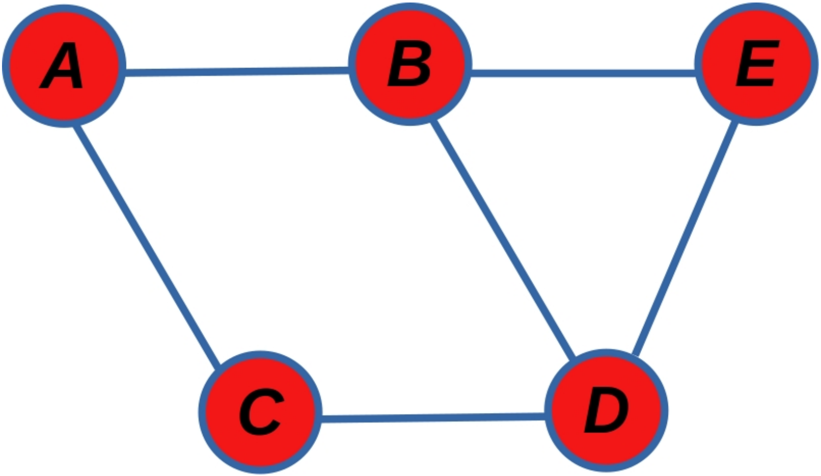
- Adjacent vertices may take same color.
- t-**improper coloring**: Max number of neighbors (adjacent nodes) with the same color is t.
- t-**improperly** k-**colorable**: Graph G has t-improper k coloring.
- t-**improper chromatic number** ($\chi^t(G)$): Least k for which Graph G is t-improperly k-colorable.
- $\chi(G) \geq \chi^t(G) \geq \chi^{t+1}(G)$, for $t \geq 0$.
- $\chi(G) = \chi^0(G)$.

[31] Kang, R.J., 2008. Improper colourings of graphs.

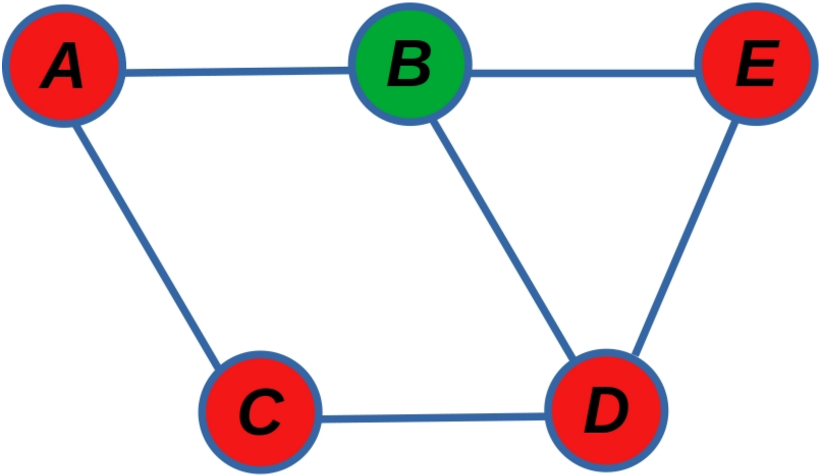
Improper Coloring



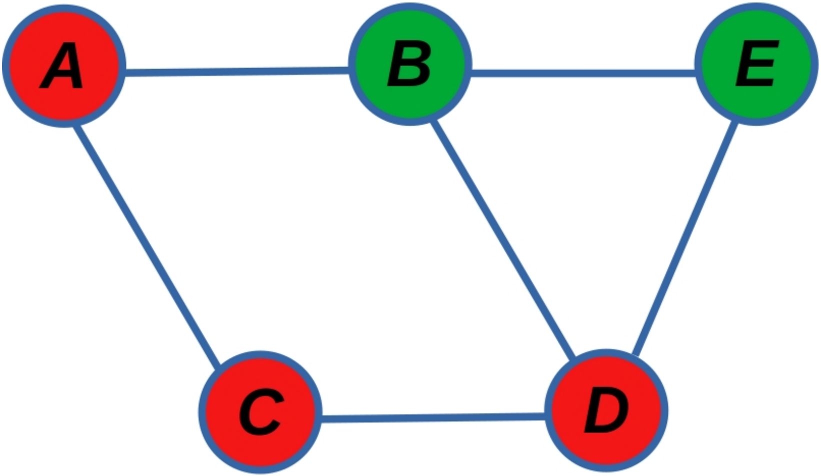
Improper Coloring: 3-improper 1-coloring, $\chi^3(G) = 1$



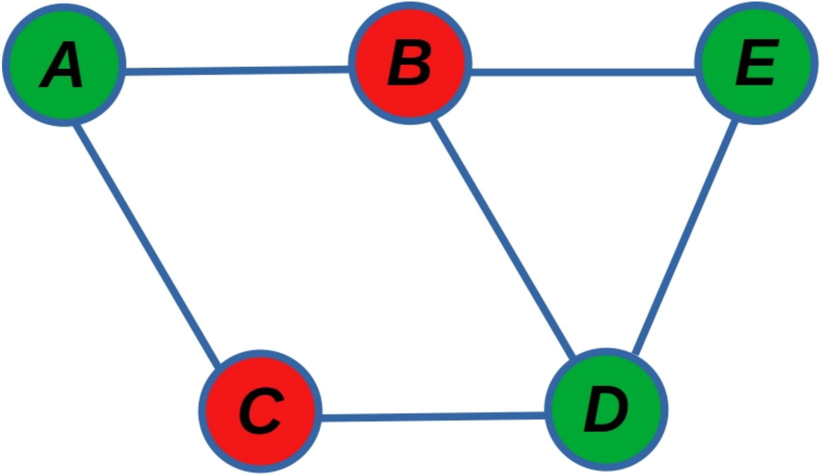
Improper Coloring: 2-improper 2-coloring, $\chi^2(G) = 2$



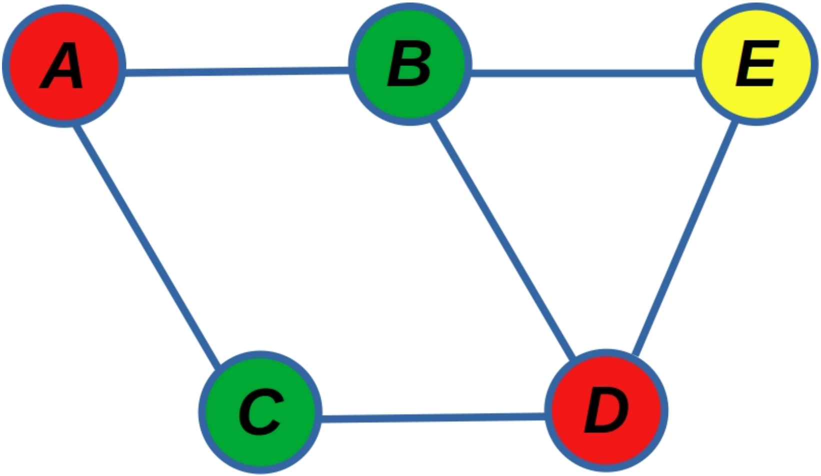
Improper Coloring: 2-improper 2-coloring, $\chi^2(G) = 2$



Improper Coloring: 1-improper 2-coloring, $\chi^1(G) = 2$



Improper Coloring: 0-improper 3-coloring, $\chi^0(G) = 3$



- **Monochromatic Component**: Component of the subgraph of G induced on vertices assigned a color.
- k**-colourable with clustering** c: Has k-coloring such that the maximum number of vertices in any monochromatic components is at most c.
- **Clustered chromatic number of a class of graphs** $(\chi^*(\mathcal{G}))$: Least k for there exist some integer c such that each graph in $\mathcal{G}$ is k-colourable with clustering c.

[32] Wood, D.R., 2018. Defective and clustered graph colouring. arXiv preprint arXiv:1803.07694.

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Thank You!