Random Metrics in the Analysis of Algorithms

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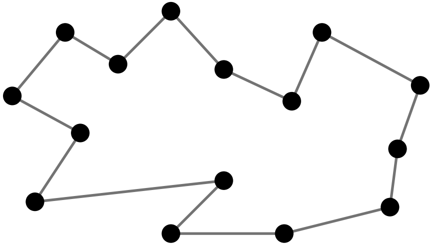
Heuristics



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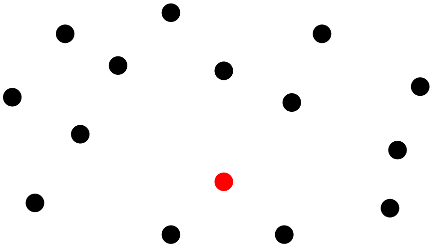

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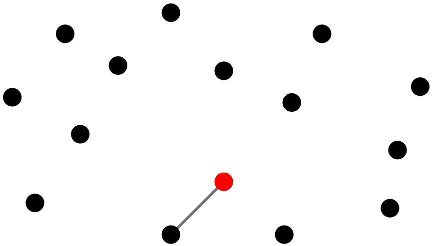

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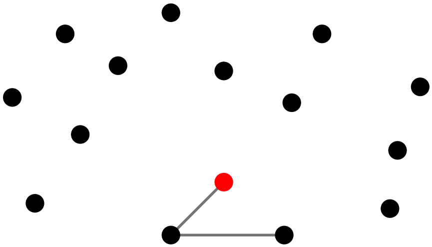

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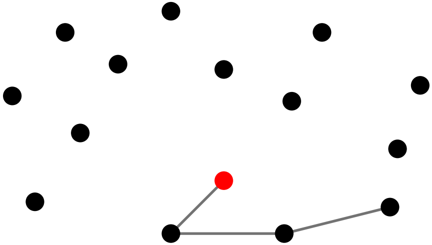

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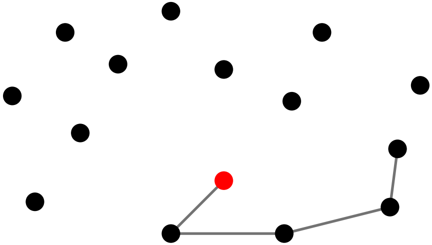

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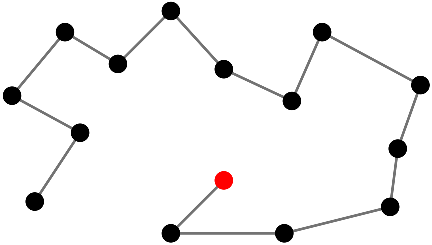

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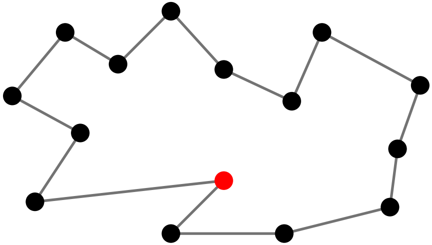

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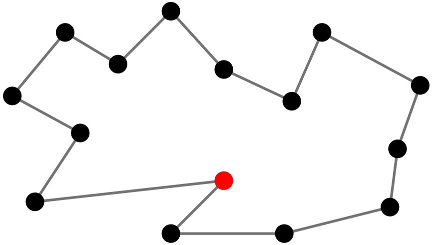
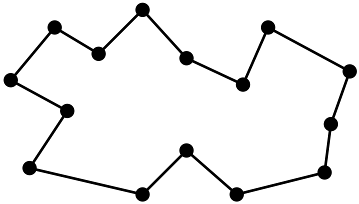
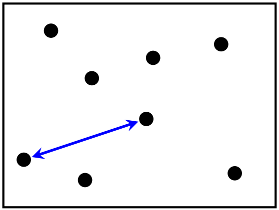


nearest neighbor for TSP:

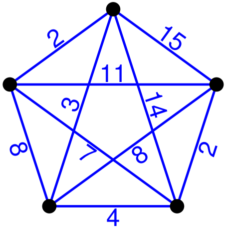
- simple construction heuristic
- worst-case approximation ratio (metric): $O(\log n)$
- experimental: ≈ 1.25
- average-case performance?

Why random metric spaces?





random in $[0, 1]^2$

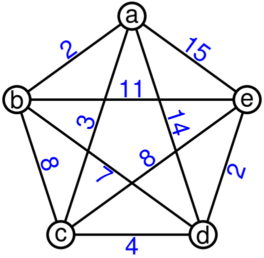


independent edge lengths

Random shortest paths = First-passage percolation

1. edge weights:
   exponentially distributed,
   independent

2. shortest paths w.r.t. weights



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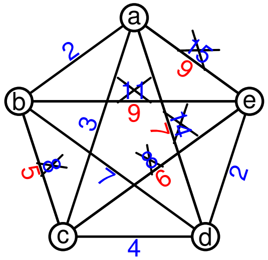


Outline

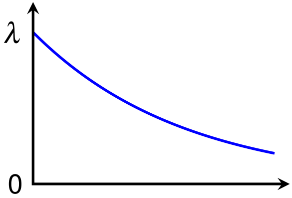
1. Properties of RSP
2. Heuristics for TSP
   - Nearest neighbor
   - Insertion heuristics
3. Facility location problem
4. General probability distributions
5. RSP with non-complete graphs
   - Random graphs
   - 2-opt on sparse graphs
6. Conclusions

Outline

Exponential distribution – Properties

Exp(λ)

- density: $\lambda e^{-\lambda x}$ for $x \geq 0$
- CDF: $1 - e^{-\lambda x}$ for $x \geq 0$
- expected value: $1/\lambda$



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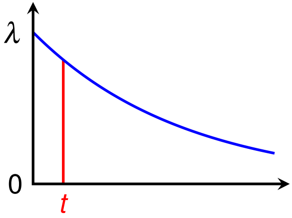

memorylessness ($R \sim \text{Exp}(\lambda)$)

$$\mathbb{P}(R \geq t + x \mid R \geq t) = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = \mathbb{P}(R \geq x)$$

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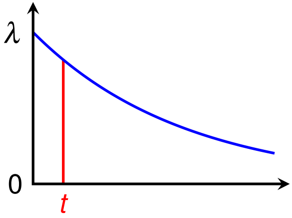

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minimum ($R_1, \ldots, R_k \sim \text{Exp}(\lambda)$; $M = \min\{R_1, \ldots, R_k\}$)

$$\mathbb{P}(M \geq x) = \prod_{i=1}^{k} \mathbb{P}(R_i \geq x) = (e^{-\lambda x})^k = e^{-(\lambda k)x}$$

$$\Rightarrow M \sim \text{Exp}(\lambda k)$$

Question

What is the typical distance in RSP?

What is $\mathbb{E}\big(d(u,v)\big)$?

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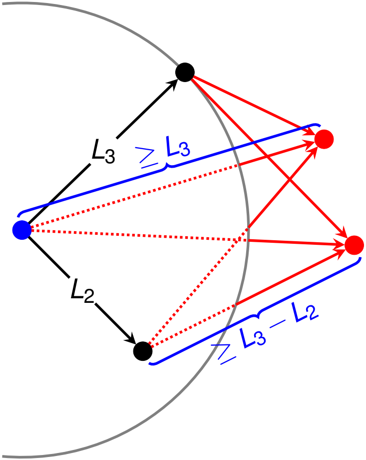
1. $\log n$
2. 1
3. $1/\sqrt{n}$
4. $\log n/n$
5. $1/n$

Distribution of RSP



$(H_n = \sum_{i=1}^{n} \frac{1}{i})$

- L_k = distance to k-th closest vertex
- $L_1 = 0$
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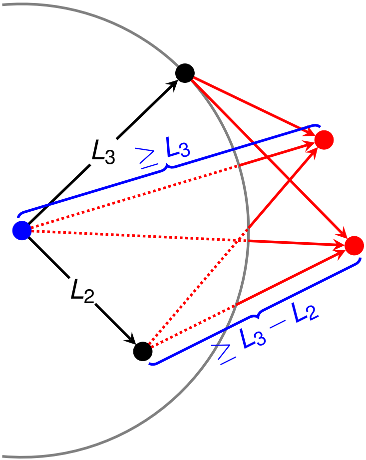

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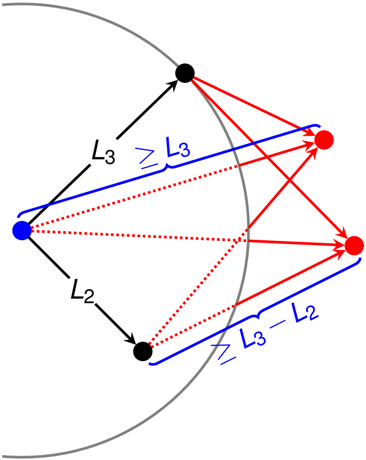

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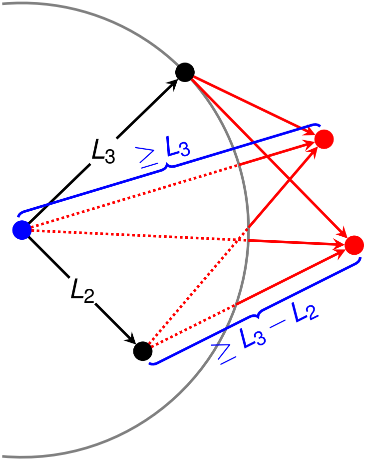

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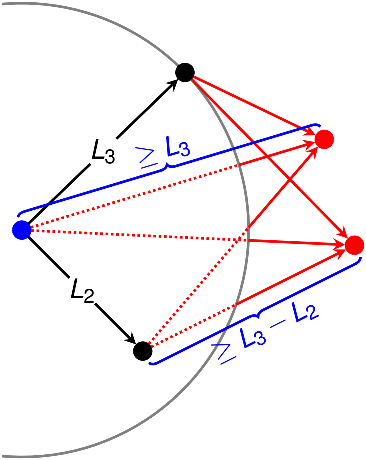

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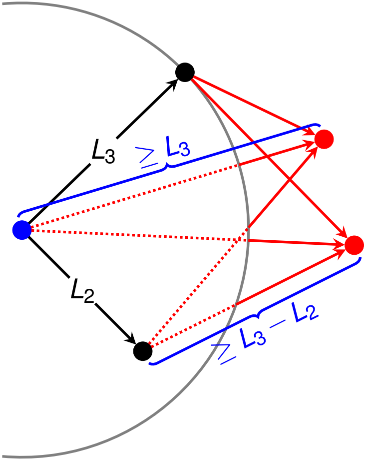

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Theorem (Janson 1999)

$$\mathbb{E}\big(\max_v d(u,v)\big) = \mathbb{E}\big(L_n\big) = 2\cdot\frac{H_{n-1}}{n} \approx 2\cdot\frac{\ln n}{n}$$

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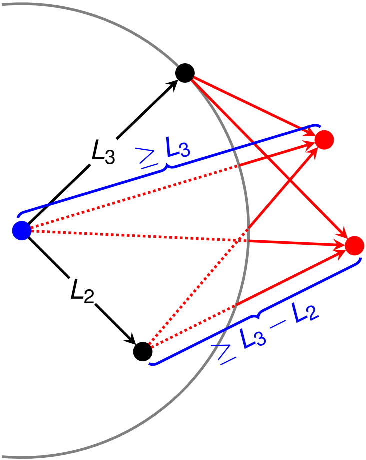

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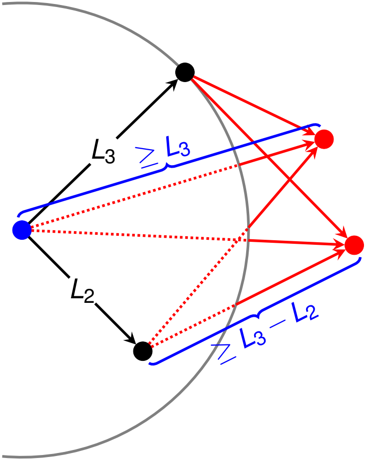

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$$\mathbb{E}\big(\max_{u,v} d(u,v)\big) \approx 3\cdot\frac{\ln n}{n}$$

Expected edge length

Theorem (Janson 1999)

$\mathbb{E}\big(d(u,v)\big) = \frac{H_{n-1}}{n-1}$

Proof.

previous slide: $\mathbb{E}(L_k) = \frac{1}{n} \cdot \big(H_{n-1} + H_{k-1} - H_{n-k}\big)$

$$\mathbb{E}\big(d(u,v)\big) = \frac{1}{n-1} \cdot \sum_{k=2}^{n} \mathbb{E}(L_k)$$

□

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□

Sums of exponential random variables

Lemma

$X \sim \sum_{i=1}^{m} \mathrm{Exp}(\lambda i)$, then $\mathbb{P}(X \leq t) = (1 - e^{-\lambda t})^m$.

Proof.

- $Y_i \sim \mathrm{Exp}(\lambda)$ independently, order statistics $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(m)}$
- $Y_{(i)} - Y_{(i-1)} \sim \mathrm{Exp}(\lambda i)$ (memorylessness)
- X has same distribution as $\max\{Y_1, \ldots, Y_m\} = Y_{(m)}$

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Lemma

$$\sum_{i=1}^{k-1} \mathrm{Exp}(ni) \leq \underbrace{\sum_{i=1}^{k-1} \mathrm{Exp}((n-i)i)}_{L_k \sim} \leq \sum_{i=1}^{k-1} \mathrm{Exp}((n-k)i)$$

Concentration of L_k

Lemma

$$\left(1-e^{-(n-k)t}\right)^k \le \mathbb{P}(L_k \le t) \le \left(1-e^{-nt}\right)^k$$

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$\square$

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□

Corollary

$\mathbb{P}(L_k > t) \leq 1 - \left(1 - e^{-(n-k)t}\right)^k \leq 1 - \left(1 - ke^{-(n-k)t}\right) = ke^{-(n-k)t}$

Proof.

$(1-x)^y \geq 1 - xy$ □

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Proof.

$(1-x)^y \ge 1 - xy$

□

Balls around nodes

Corollary

$\mathbb{P}(L_k > t) \leq ke^{-(n-k)t}$

ball around v:

$B_t(v) = \{u \in V \mid d(v,u) \leq t\}$

Corollary

$\mathbb{P}\big(|B_t(v)| < k\big) = \mathbb{P}\big(L_k > t\big) \leq ke^{-(n-k)t}$

Global structure

Lemma

"$\mathbb{P}\big(v \text{ has} < e^{nt} \text{ neighbors within distance } t\big) \leq e^{-nt}$"

example: $$\mathbb{P}\left(v \text{ has} < \log n \text{ neighbors within } \frac{\log\log n}{n}\right) \leq \frac{1}{\log n}$$

Global structure

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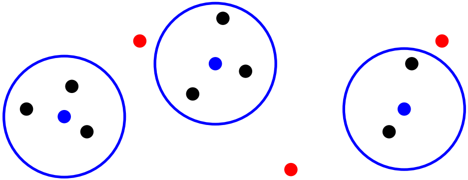
"$\mathbb{P}\big(v$ has $< e^{nt}$ neighbors within distance $t\big) \leq e^{-nt}$"



$\frac{n}{e^{nt}}$ components of size e^{nt} and diameter $O(t)$

$\frac{n}{e^{nt}}$ orphans

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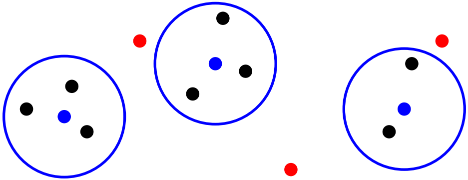

$\frac{n}{e^{nt}}$ components of size e^{nt} and diameter $O(t)$

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is this really possible without leftover vertices?

Global structure

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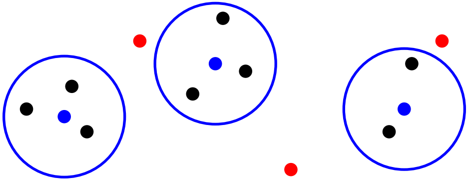

- greedily pick vertex v with $B_t(v) \geq e^{nt}$ without marked neighbor
- mark all vertices in $B_t(v)$ and keep going
- assign remaining vertices to some cluster or keep them as orphans

Outline

1. Properties of RSP
2. Heuristics for TSP
   - Nearest neighbor
   - Insertion heuristics
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4. General probability distributions
5. RSP with non-complete graphs
   - Random graphs
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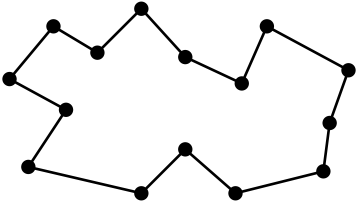
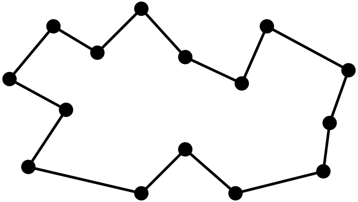
Optimal TSP tour



Theorem

$\mathbb{E}(\text{length of optimal TSP tour}) = \Omega(1)$ (in fact, $\Theta(1)$)

Optimal TSP tour



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$\mathbb{E}$(length of optimal TSP tour) $= \Omega(1)$ (in fact, $\Theta(1)$)

- length of shortest edge at v does not change under shortest path
- length of shortest edge at $v \sim \text{Exp}(n-1)$
- $\mathbb{E}$(length of shortest edge at v) $= \frac{1}{n-1}$
- linearity of expectation

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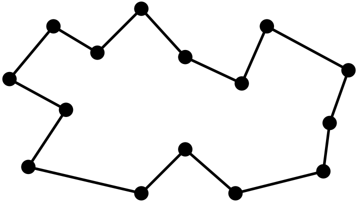

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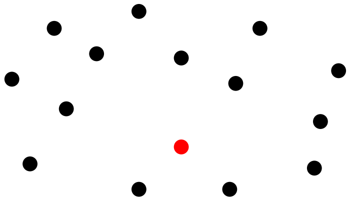
- linearity of expectation
- holds even without RSP

Nearest neighbor for TSP



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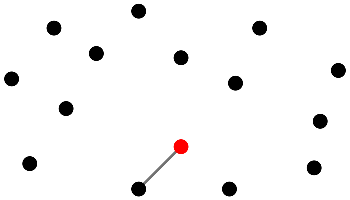

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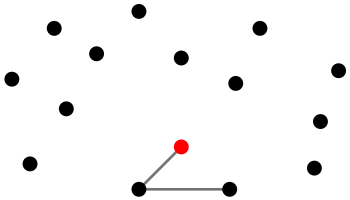

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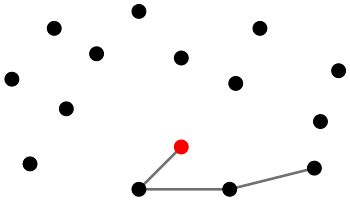

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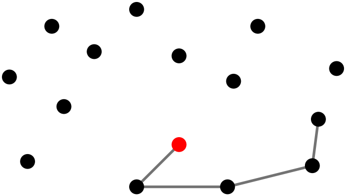

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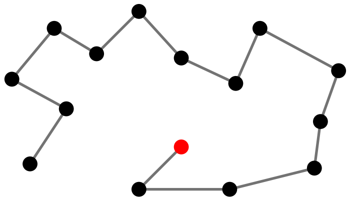

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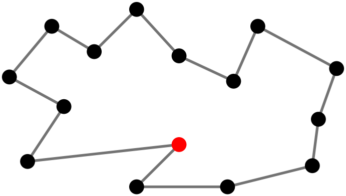
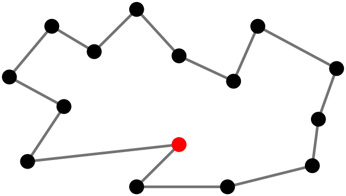


Average-case nearest neighbor without RSP



What is the expected tour length?

distances independent, Exp(1), no RSP, no triangle inequality

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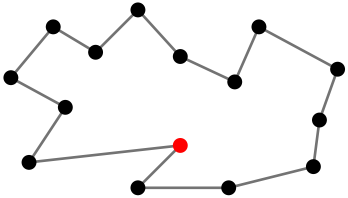

What is the expected tour length?

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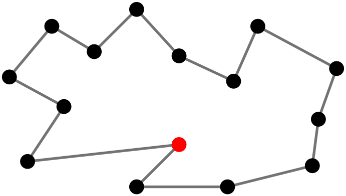
1. 1
2. $\log n$
3. $\sqrt{n}$
4. n

Average-case nearest neighbor without RSP



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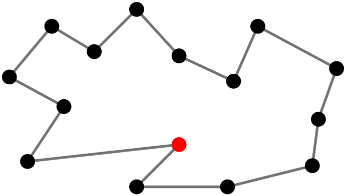

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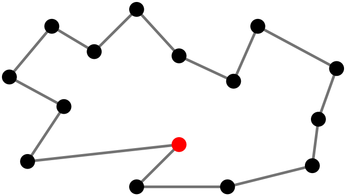

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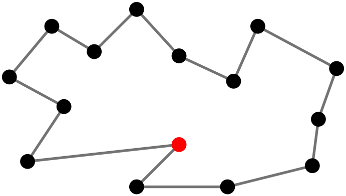

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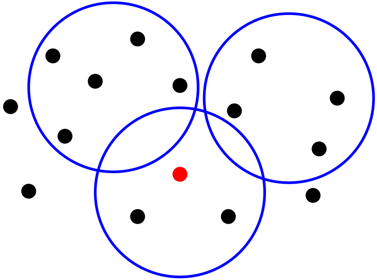
► $\mathbb{E}(\text{NN without RSP}) = H_{n-1} + \frac{n}{n-1} = \Theta(\log n)$

Nearest neighbor for TSP



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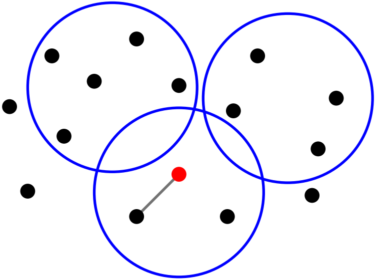

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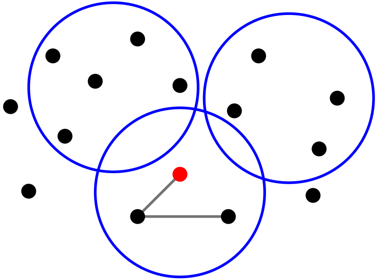

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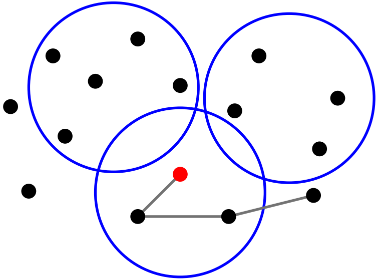

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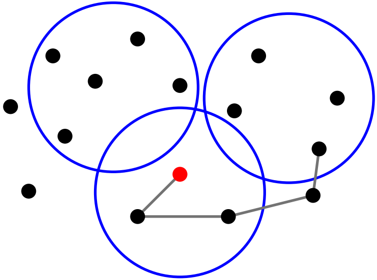

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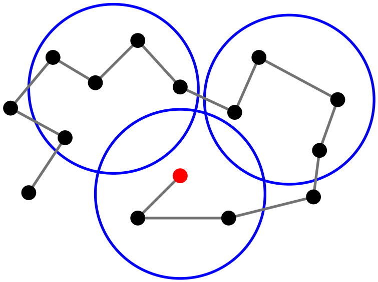

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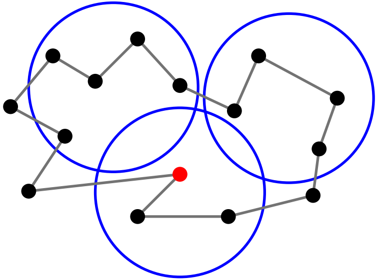
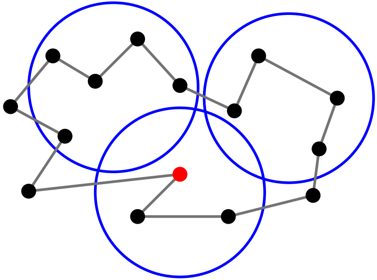


- $\leq$ 1 long outgoing edge per component/orphan

Nearest neighbor for TSP



- ≤ 1 long outgoing edge per component/orphan
- $t_i = \frac{i}{n}$ for $i = 1, 2, 3, \ldots$

  long edges for phase i are estimated as $\leq t_{i+1}$

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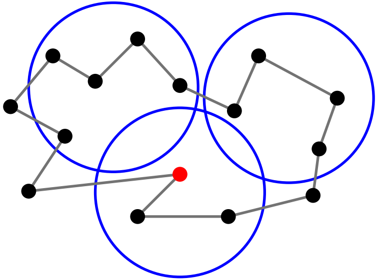

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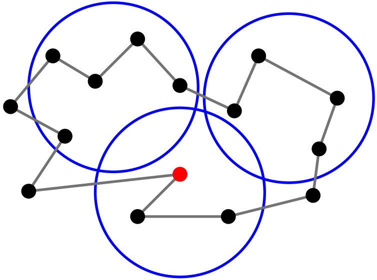

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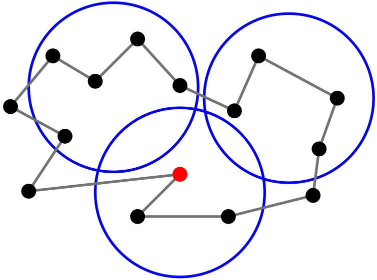

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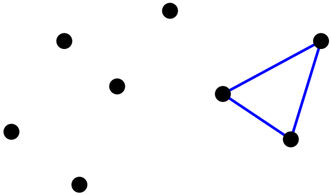
Theorem

nearest neighbor: expected length $O(1)$, expected approximation ratio $O(1)$

Insertion heuristics



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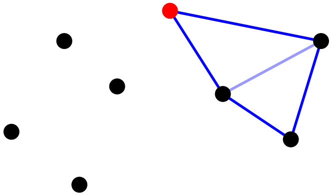

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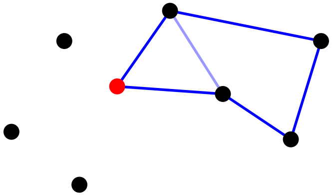

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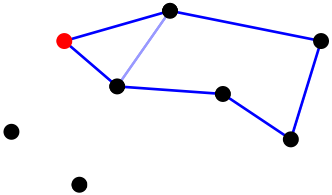

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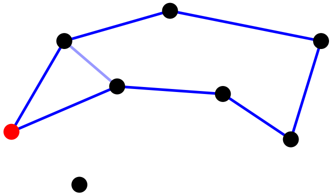

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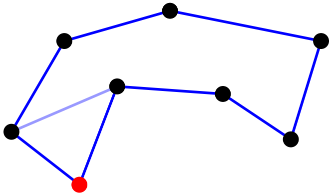

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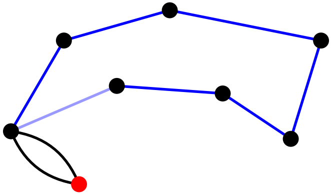

- every component: cheap insertion from second point on
- costs $> t$ only once per component

Theorem

every insertion heuristic achieves expected ratio $O(1)$

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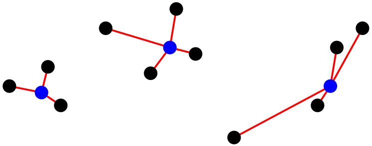
k-center

- find $C \subseteq V$ with $|C| = k$
- minimize $\sum_{v \in V} \min_{c \in C} \big(d(v, c) \big)$



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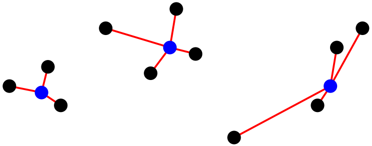

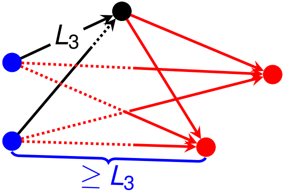

- fixed C
- $L_{k+1}, \ldots, L_n$: distances to C
- $L_1, \ldots, L_k = 0$
- $t_j = L_{j+1} - L_j = \mathrm{Exp}\big(j \cdot (n-j)\big)$

$$\text{costs} = \sum_{i=k+1}^{n} L_i = \sum_{j=k}^{n-1} (n-j) \cdot t_j = \sum_{j=k}^{n-1} \mathrm{Exp}\big(j\big)$$

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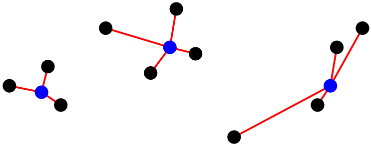

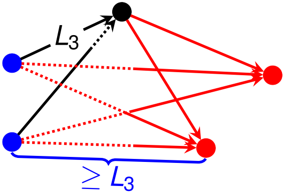

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Theorem

any solution is a $\big(1 + o(1)\big)$-approximation for $k = O(n^{1-\varepsilon})$

Facility location problem

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- minimize $\sum_{v \in C} f_v + \sum_{v \in V} \min_{c \in C} \big(d(v, c) \big)$

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- opening k cheapest facilities: $\mathbb{E}(\text{costs}) = \sum_{i=1}^{k} f_i + \sum_{i=k}^{n-1} \frac{1}{i}$

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- more challenging: lower bounds for optimal solution
- Cauchy–Schwarz for "decoupling":

$$\mathbb{E}\left(\frac{\text{ALG}}{\text{OPT}}\right) \leq \sqrt{\mathbb{E}(\text{ALG}^2) \cdot \mathbb{E}\left(\frac{1}{\text{OPT}^2}\right)}$$

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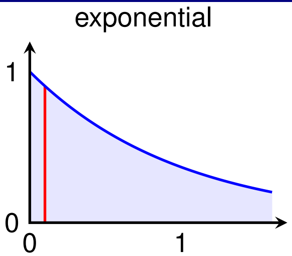
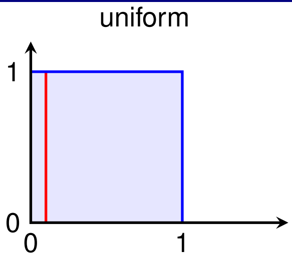
Theorem

ALG is a $\big(1 + o(1)\big)$-approximation for $k = o(n)$

Outline

General probability distributions

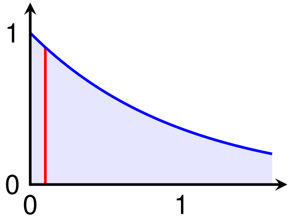




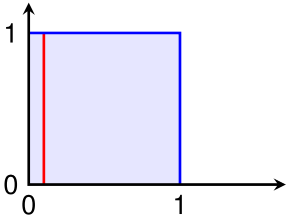
can we transfer the results to uniform/arbitrary distributions?

General probability distributions

exponential



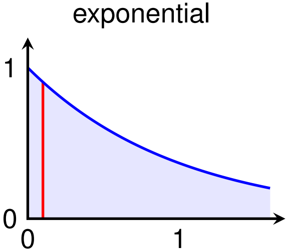
uniform



can we transfer the results to uniform/arbitrary distributions?

- observation: distances decrease with n

General probability distributions



can we transfer the results to uniform/arbitrary distributions?

- observation: distances decrease with n
- density f: differentiable in $(0, \varepsilon)$
- $\mathbb{P}(\text{weight} \leq x) = x + o(x)$
  (every distribution is approximately uniform in $(0, \varepsilon)$)

► results carry over, scale by $1 \pm o(1)$

Outline

RSP generated from non-complete graphs

- balls grow extremely fast: $\left|B_t(v)\right| = \left|\left\{u \in V \mid d(v,u) \leq t\right\}\right| \approx e^{nt}$
- unbounded doubling dimension

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- RSP metrics can be generated from any connected graph

  (the metric is always complete!)

► use non-complete/sparse graphs!

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  (the metric is always complete!)

► use non-complete/sparse graphs!

- challenges:
  - unknown structure
  - lack of symmetry

RSP with $G_{n,p}$ random graphs

$G_{n,p}$

- n vertices
- $\mathbb{P}\big(\{u, v\} \in E\big) = p$ independently

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1. draw random graph – connected w.h.p.
2. draw edge weights for existing edges
3. shortest paths

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can we reuse results for complete graphs?

RSP with $G_{n,p}$ – coupling

$$
\begin{aligned}
& G_{n,p} \text{ with } \mathrm{Exp}(1) \\
\approx\ & G_{n,p} \text{ with } U(0,1) \\
=\ & \text{complete graph with } U(0,1/p)\text{, remove edges of weight} \geq 1 \\
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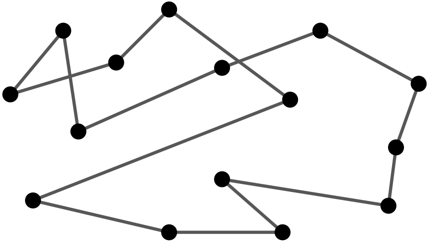
all “sum of lengths” results scale by $1/p$

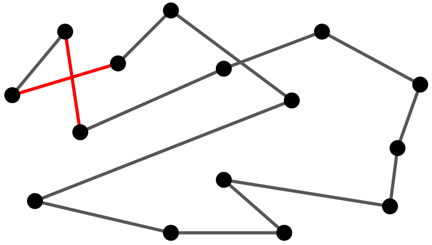
Theorem

“approximation results for complete graphs also hold for $G_{n,p}$”

2-opt heuristic for TSP



2-opt heuristic for TSP



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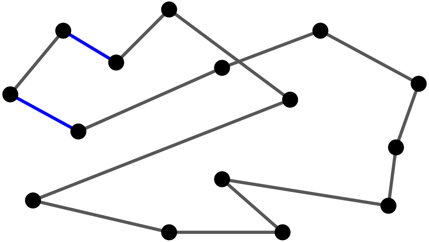

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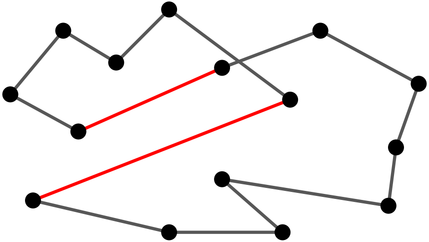

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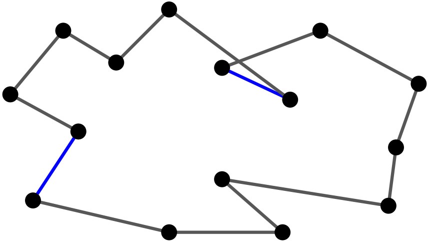

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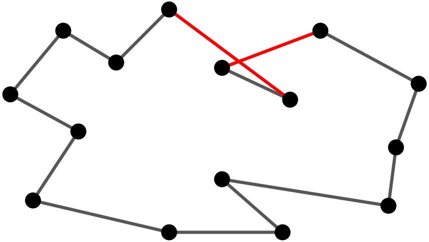

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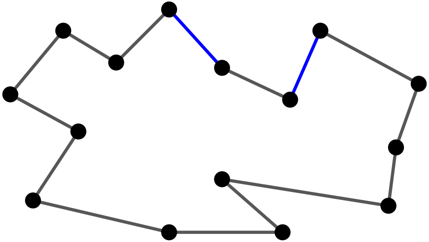

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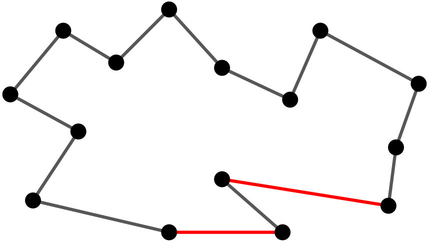

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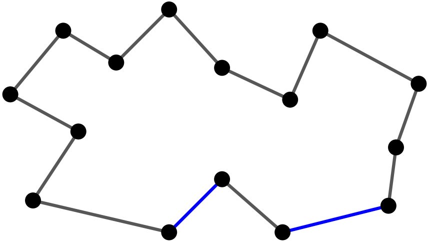
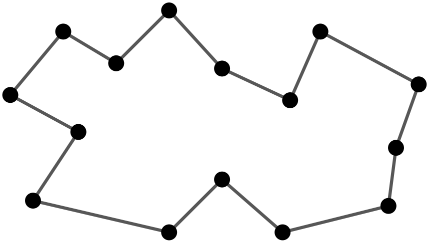


2-opt heuristic for TSP – approximation ratio

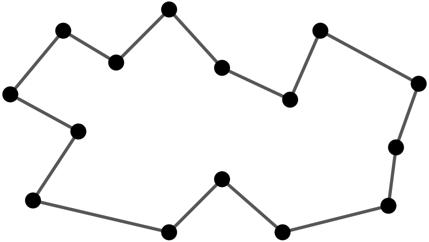


worst-case: $O(\sqrt{n})$

RSP on complete graphs: $O(\log n) \rightsquigarrow$ trivial – why?

RSP on sparse graph: $O(1) \rightsquigarrow$ now

2-opt heuristic for TSP – approximation ratio



worst-case: $O(\sqrt{n})$

RSP on complete graphs: $O(\log n) \rightsquigarrow$ trivial – why?

maximum edge $\approx 3 \cdot \frac{\ln n}{n}$ & optimal tour $= \Omega(1)$

RSP on sparse graph: $O(1) \rightsquigarrow$ now

2-opt – RSP in sparse graphs

- sparse graph: $m = \Theta(n)$ edges
- S_k: sum of k lightest edge weights
- $S_n \leq \text{TSP}$

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- $S_n \leq \text{TSP}$

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- ► $\text{TSP} = \Omega(n)$

2-opt – RSP in sparse graphs

Lemma

$\mathbb{E}(S_k) = \Omega(k^2/n)$

Proof.

- $w_1, w_2, \ldots$: edge weigths in increasing order
- $w_1 \sim \mathrm{Exp}(m)$
- $w_{i+1} - w_i \sim \mathrm{Exp}(m-i) \rightsquigarrow w_i \sim \sum_{j=0}^{i-1} \mathrm{Exp}(m-j)$
- $S_k = \sum_{i=1}^{k} w_i \sim \sum_{j=0}^{k-1} (k-j)\, \mathrm{Exp}(m-j) = \sum_{j=0}^{k-1} \mathrm{Exp}\left(\frac{m-j}{k-j}\right)$

  $\geq \sum_{j=0}^{k-1} \mathrm{Exp}\left(\frac{m}{k}\right)$
- $\mathbb{E}(S_k) \geq \frac{k^2}{m}$

□

Approximation ratio of 2-opt in sparse graphs

- $P_{(u,v)} =$ edges of original graph on shortest u–v path
- edges are considered directed for $P_{(u,v)}$

Lemma

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- upper bound for 2-optimal tour: twice the sum of edge weights
- constant fraction of all edge weights appears in optimal tour
- 2-optimal tours have length $O(n)$
  global optimum has length $\Omega(n)$

Theorem

2-opt on sparse graphs achieves approximation ratio $O(1)$

Outline

Summary & open problems

summary

- RSP models random metrics
- TSP: nearest neighbor, insertion, 2-opt
- trivial algorithm for facility location & k-center
  (consider as concentration of measure)
- some first results on incomplete graphs

open problems

- more for sparse graphs
- directed graphs
- other models for random metrics?

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